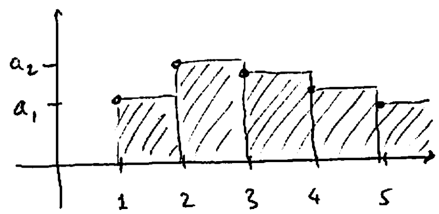
10/21/2011

Integral Test (§15)

6 ~~Wacom~~ Consider a series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$. Geometric viewpoint:

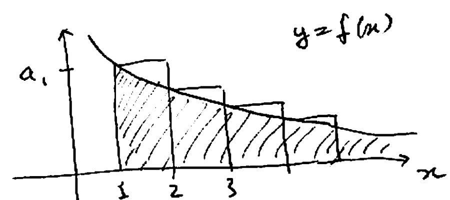
~~Look at~~ Notice: $S = \sum_{n=1}^{\infty} a_n$ is the total area of the rectangles:

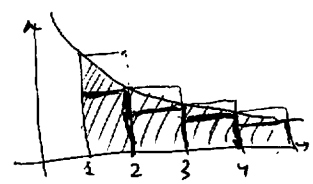


- Such areas can be computed ~~by~~ integration

~~Conv~~ Convert the sequence a_n into a function ~~f(x)~~ $f(x)$ such that $a_n = f(n)$.

Suppose $f(x)$ decreases.



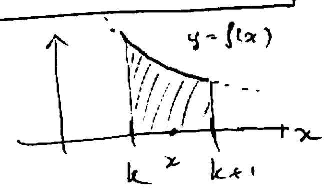


$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n$; $\quad \int_1^{\infty} f(x) \geq \sum_{n=2}^{\infty} a_n$

Hence $\sum_{n=1}^{\infty} a_n$ converges iff $\int_1^{\infty} f(x)\,dx < \infty$

THM (Integral test). Suppose $a_n = f(n)$, where $f(x)$ is a positive non-increasing function on $[1,\infty)$. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges iff } \int_1^{\infty} f(x)\,dx < \infty.$$



Proof Compare $S_n = \sum_{k=1}^{n} a_k$ and $t_n = \int_1^n f(x)\,dx$.

Since $f(x)$ is non-increasing,

$$\underset{a_{k+1}}{f(k+1)} \leq f(x) \leq \underset{a_k}{f(k)} \quad \text{for } x \in [k, k+1].$$

Integration yields $\quad a_{k+1} \leq \int_k^{k+1} f(x)\,dx \leq a_k$

Sum these inequalities over $k = 1, \dots, n-1$: for each n.

$$\underbrace{\sum_{k=1}^{n-1} a_{k+1}}_{=\,a_2+a_3+\dots+a_n\,=\,S_n - a_1} \leq \underbrace{\int_1^n f(x)\,dx}_{=\,t_n} \leq \underbrace{\sum_{k=1}^{n-1} a_k}_{=\,S_{n-1}}$$

Hence (S_n) is bounded if and only if (t_n) is. Since $f(x) \geq 0$, both (s_n) and (t_n) are non-decreasing. So by Monotone Convergence Thm, (s_n) converges iff (t_n) converges.

QED

Example (a). $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

Integral test: $\int_1^{\infty} \frac{1}{x}\,dx = \ln x \Big|_1^{\infty} = \infty$. QED.

(b). $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \le 1$.

(Integral test: $\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_1^{\infty}$ (for $p \neq 1$)

$$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p<1 \end{cases}$$).

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = +\infty$

$$\left(\int_2^{\infty} \frac{dx}{x \ln x} = \left\{ \begin{array}{l} \ln x = y \\ \frac{dx}{x} = dy \end{array} \right\} = \int_{\ln 2}^{\infty} \frac{dy}{y} = \infty \right)$$

(d) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} < \infty$

$$\left(\int_2^{\infty} \frac{dx}{x \ln^2 x} = \int_{\ln 2}^{\infty} \frac{dy}{y^2} < \infty \right)$$

Thm (Limit Comparison Test) Suppose that $a_n \ge 0$, $b_n > 0$ and $r = \lim \frac{a_n}{b_n}$.

(i) If $0<r<\infty$ then $\sum a_n$ converges if and only if $\sum b_n$ converges

(ii) If $r=0$ and $\sum b_n$ converges then $\sum a_n$ converges

(iii) If $r=\infty$ and $\sum b_n = \infty$ then $\sum a_n = \infty$

Proof (i) let $0<r<\infty$. By def. of limit (for $\varepsilon = r/2$), there exists N such that

$$\left| \frac{a_n}{b_n} - r \right| < \frac{r}{2} \quad \text{for } n>N$$

$$\frac{1}{2} b_n < a_n < \frac{3}{2} b_n \quad \text{for } n>N.$$

Then the conclusion follows from the Comparison Test (Thm 14.6)

(ii), (iii) – similar (Exercise). QED.

(15.3)

Thm (Alternating Series Theorem).

> If $a_n \ge 0$, (a_n) is non-increasing and $\lim a_n = 0$ then $\sum (-1)^n a_n$ converges.

Proof We will use Cauchy test, where we look at ~~$n > m$ and ε~~ $\sum_{k=m}^{n} (-1)^k a_k$ for $n > m$ large, and want to show this is small.

~~$S_{n,m} = a_m \; a_{m+}$~~

Up to sign, $\sum_{k=m}^{n} (-1) a_k$ equals

$$S_{n,m} := \underbrace{(a_m - a_{m+1})}_{\ge 0} + \underbrace{(a_{m+2} - a_{m+3})}_{\ge 0} + \underbrace{(a_{m+4} - a_{m+5})}_{\ge 0} + \dots + \underbrace{(a_{n-1} - a_n)}_{\ge 0} \ge 0 \quad \text{if } n-m \text{ is } \textbf{odd}$$

On the other hand, $$S_{n,m} = a_m - \underbrace{(a_{m+1} - a_{m+2})}_{\ge 0} + \underbrace{(a_{m+3} - a_{m+4})}_{\ge 0} + \underbrace{(a_{m+5} - a_{m+6})}_{\ge 0} + \dots + \underbrace{(a_{n-2} - a_{n-1})}_{\ge 0} - a_n$$

$$\le a_m - a_n \le a_m.$$

So $$0 \le S_{n,m} \le a_m \qquad \text{for } n-m \text{ odd}.$$

If $n-m$ is even, $n-(m-1)$ is odd, and $S_{n,m} = S_{n,m-1} \pm a_m$, so

$$0 \le S_{n,m} \le a_{m-1} \pm a_m \le 2a_{m-1}.$$

Hence in either case (since $a_m \le a_{m-1}$) we have

$$0 \le S_{n,m} \le 2a_{m-1}.$$

Since $a_{m-1} \to 0$ as $m \to \infty$, we conclude that Cauchy Test holds:

> For every $\varepsilon > 0$ there exists N such that
> $$\left| \sum_{k=m}^{n} (-1)^k a_k \right| = |S_{n,m}| < \varepsilon \quad \text{for } n > m > N.$$

(Why? Make $2a_{m-1} < \varepsilon$).

Therefore $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. **QED**

Example: $\sum \frac{(-1)^n}{n}$ converges by A.S.T.

Recall that $\sum \frac{1}{n}$ diverges (harmonic series).

Therefore, $\sum \frac{(-1)^n}{n}$ converges but not absolutely.