§20. Limits Of Functions.

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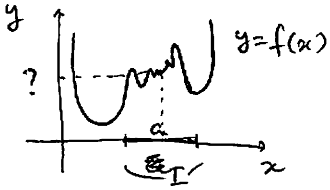


- $\lim_{x\to a} f(x)$ describes the <u>local behavior</u> of f near a.
- f needs not even be defined <u>at</u> a.

Def Let $a \in \mathbb{R}$; I be an open interval $\ni a$. consider a function f defined ~~on an open interva~~ on $I \setminus \{a\}$ ~~everywhere on I except possibly at a~~.

We say "f(x) converges to L as x converges to a" and write

$$\lim_{x\to a} f(x) = L \qquad (\text{or, equivalently, } f(x) \to L \text{ as } x \to a)$$

if, for every sequence (x_n) in $I \setminus \{a\}$,

$$x_n \to a \text{ implies } f(x_n) \to L.$$

<u>Remark</u>: f is <u>continuous</u> at a $\iff$ $\lim_{x\to a} f(x) = f(a)$.

<u>Examples</u>.

(a) $\lim_{x\to 2} x^2 - x + 2 = 2^2 - 2 + 2 = 4$ by continuity of $f(x) = x^2 - x + 2$.

(b) $$\lim_{x\to 0} \frac{\sqrt{a+x} - \sqrt{a}}{x} = \lim_{x\to 0} \frac{x}{x(\sqrt{a+x} + \sqrt{a})} = \lim_{x\to 0} \frac{1}{\sqrt{a+x} + \sqrt{a}}$$

Let $x_n \to 0$.

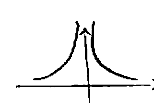
$$\lim \frac{1}{\sqrt{a+x_n} + \sqrt{a}} = \frac{1}{\sqrt{a+0} + \sqrt{a}} \quad (\text{by limit theorems})$$

$$= \frac{1}{2\sqrt{a}}.$$

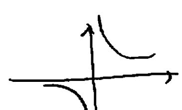
Hence $$\boxed{\lim_{x\to 0} \frac{\sqrt{a+x} - \sqrt{a}}{x} = \frac{1}{2\sqrt{a}}.}$$

<u>Remark</u> If $L = \pm\infty$ the def of "divergence to $\pm\infty$" is similar.

<u>Ex</u> (c) $\lim_{x\to 0} \frac{1}{x^2} = \infty$; $\left(\lim x_n \to 0 \text{ implies } \lim \frac{1}{x_n} = \infty\right)$



(d) $\lim_{x\to 0} \frac{1}{x}$ DNE



$$\left(\begin{array}{l} x_n = \frac{1}{n} \Rightarrow \lim \frac{1}{x_n} = \infty \\ x_n = -\frac{1}{n} \Rightarrow \lim \frac{1}{x_n} = -\infty \end{array}\right).$$

Thm 20.6 (ε-δ def of limit) [Cauchy]

Let $a \in \mathbb{R}$, $I \subset \mathbb{R}$ open interval containing a, f defined on $I \setminus \{a\}$, $L \in \mathbb{R}$.

Then
$$\lim_{x \to a} f(x) = L \quad \text{if and only if:}$$

$$(*)\left\{ \begin{array}{l} \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \\ \qquad |x-a| < \delta,\ x \in S \ \text{ implies } \ |f(x) - L| < \varepsilon \end{array} \right.$$

(Compare to ε-δ def.[17.2] of continuity, where $L = f(a)$).

($\Leftarrow$) Assume ~~$\lim f(x) = L$~~ $(*)$ holds, consider (x_n): $\lim x_n = a$. WTS: $\lim f(x_n) = L$.

~~$\forall \varepsilon > 0 \ \exists$~~ Let $\varepsilon > 0$. Choose δ as in $(*)$

By , $\exists N$: $|x_n - a| < \delta$ for $n > N$.

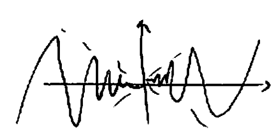
By $(*)$, $|f(x_n) - L| < \varepsilon$ for $n > N$. QED

($\Rightarrow$) Assume $\lim_{x \to a} f(x) = L$ but $(*)$ fails,

i.e. $\exists \varepsilon > 0$ s.t. $\exists x_n$: $\underbrace{|x_n - a| < \frac{1}{n}}_{\Downarrow \ \lim x_n = a}$ but $\underbrace{|f(x_n) - L| \geq \varepsilon}_{\Downarrow \ \lim f(x_n) \neq L}$.

Hence $\lim_{x \to a} f(x) \neq L$. QED ↯.

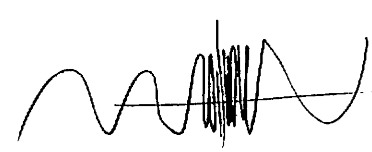
Examples (e). $\lim_{x \to 0} x \sin \frac{1}{x} = 0$



: $|x| < \delta$ implies $|x \sin \frac{1}{x}| \leq |x| \cdot 1 \leq \delta$.

Hence def $(*)$ holds with $\varepsilon = \delta$.

(f) $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist (~~Ex choose~~ (why?))



Thm 20.4 (Operations). Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, $L, M \in \mathbb{R}$. Then:

(i) $\lim_{x\to a}(f+g)(x) = L+M$;

(ii) $\lim_{x\to a}(fg)(x) = LM$;

(iii) $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = L/M$ provided that $M \neq 0$ ~~and $g(x) \neq 0$~~

(iv) $\lim_{x\to a}(kf)(x) = kL$ $\forall k \in \mathbb{R}$.

[Analogous to Thm 17.4 for operations on continuous functions - use sequential def of limit]

Thm 20.5 (Composition) Let $\lim_{x\to a} f(x) = L \in \mathbb{R}$.

Let g be a continuous function on some open interval containing L.

Then $\lim_{x\to a}(g\circ f)(x) = \lim_{x\to a} g(f(x)) = g(L)$.

[Analogous to Thm 18.5 on composition of continuous functions]

Ex (g) $\lim_{x\to 0} \frac{x^2 - \cos x}{2+\tan x} = \frac{0^2 - \cos(0)}{2+\tan(0)} = -\frac{1}{2}$ by limit thms, using continuity of $\cos x, \tan x$ at 0.

Thm (Squeeze Thm) Let f, g, h be defined on an open interval I containing a (except possibly at a).

If $g(x) \le h(x) \le f(x)$ for all $x \in I\setminus\{a\}$

then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = L$ implies $\lim_{x\to a} h(x) = L$.

$L \in \mathbb{R} \cup \{\pm\infty\}$

Thm (Comparison) If $f(x) \le g(x)$, $x \in I\setminus\{a\}$ and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist $\in \mathbb{R}$,

then $\lim f(x) \le \lim g(x)$.

Ex (h) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$: Use the inequality $x\cos^2 x \le \sin x \le x$ for $|x| < \pi/2$. (Prove it!)

$$\underset{1-\sin^2 x}{\overset{\cos^2 x}{\shortparallel}} \le \frac{\sin x}{x} \le 1. \quad \text{Squeeze Thm:} \left\{ \begin{array}{l} \lim_{x\to 0} 1 = 1, \\ \lim_{x\to 0}(1-\sin^2 x) = 1 - \left(\lim_{x\to 0} \sin x\right)^2 = 1-0 = 1 \\ \qquad \text{(by continuity).} \end{array} \right.$$

$$\Downarrow$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$