# One-sided limits

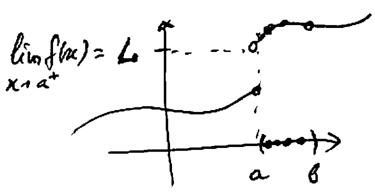


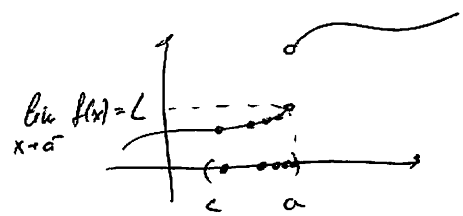
**Def** (One-sided limits)

- Let $a \in \mathbb{R}$, $L \in \mathbb{R}$. ~~and consider a function defined on some open interval $(a,b)$~~.

  We say: "$f(x)$ converges to $L$ as $x$ approaches $a$ <u>from the right</u>" and write
  $$\lim_{x \to a^+} f(x) = L$$
  if $f$ is defined on some open interval $(a,b)$ and for every sequence $(x_n)$ in $(a,b)$,
  $$x_n \to a \text{ implies } f(x_n) \to L.$$

- Similarly, "$f(x)$ converges to $L$ as $x \to a$ <u>from the left</u>" and write
  $$\lim_{x \to a^-} f(x) = L$$
  if $f$ is defined on some open interval $(c, a)$ and, for every $(x_n)$ in $(c, a)$
  $$x_n \to a \text{ implies } f(x_n) \to L.$$




**Thm** ($\varepsilon$-$\delta$ definition) – analogous to Thm 20.6

- Let $a \in \mathbb{R}$, $L \in \mathbb{R}$. Then

  $f$ is defined on some $(a, b)$

  $$\lim_{x \to a^+} f(x) = L \quad \text{if and only if:}$$

  $\forall \varepsilon > 0 \ \exists \delta > 0$ such that
  $$\underbrace{a < x < a + \delta}_{(a+\delta < b)} \text{ implies } |f(x) - L| < \varepsilon.$$

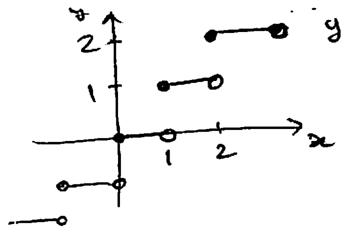
- Similarly for $\lim_{x \to a^-} f(x) = L$: change to $\underbrace{a - \delta < x < a}_{(a-\delta > c)}$.

**Examples** (a) $\lim_{x\to 0}\sqrt{x}$ does not exist ( $\sqrt{x}$ is not defined for $x<0$ ) but $\lim_{x\to 0^+}\sqrt{x} = 0$.

( $0 < x < \varepsilon$ implies $0 < \sqrt{x} < \sqrt{\varepsilon}$ $\Rightarrow$ $|\sqrt{x}| < \sqrt{\varepsilon}$.

$\Rightarrow$ Def. holds with $\delta = \varepsilon$. )

(b) $[x]$ = integer part of $x$ := largest integer $\le x$.

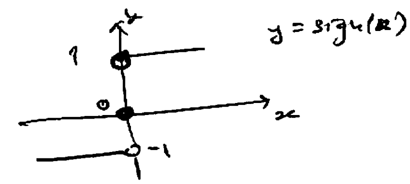



$\forall n \in \mathbb{Z}$, $\lim_{x\to n}[x]$ does not exist, but

$\lim_{x\to n^+}[x] = n$, $\lim_{x\to n^-}[x] = n-1$. (Ex).

$[x]$ is "right-continuous" ( $\lim_{x\to x_0^+} f(x) = f(x_0)$ )

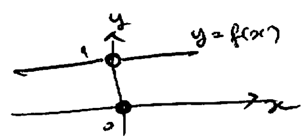
(c) The sign function $\quad \operatorname{sign}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$




$\lim_{x\to 0}\operatorname{sign}(x)$ does not exist, but

$\lim_{x\to 0^+}\operatorname{sign}(x) = 1$, $\quad \operatorname{sign}_{x\to 0^-} \operatorname{sign}(x) = -1$ $\quad$ ( Ex )

(d) $f(x) = |\operatorname{sign}(x)|$




$\underbrace{\lim_{x\to 0} f(x) = 1}$ although $f(0) = 0$, so $f(x)$ is not continuous at $0$.

↑ NOTE THIS!

**Proposition** (Thm 20.10) Let $f$ be defined in an open interval $I$ containing $a$ except possibly at $a$.

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

**Proof** ($\Rightarrow$) – obvious (Ex).

($\Leftarrow$) If one-sided limits $= L$, then $\forall \varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$a < x < a+\delta_1$, implies $|f(x) - L| < \varepsilon$

$a - \delta_2 < x < a$ implies the same.

Hence: $|x-a| < \delta = \min(\delta_1, \delta_2)$, $x \ne a$ implies $|f(x) - L| < \varepsilon$.

So $\lim_{x\to a} f(x) = L$. $\quad$ QED

Ex: in (d), $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = 1$

# Limits at infinity. (§20)

**Def** (limits at $\infty$)

- Let $L \in \mathbb{R}$. We say "$f(x)$ converges to $L$ as $x \to +\infty$ (respectively, $-\infty$)" and write $\lim_{x \to \pm\infty} f(x) = L$ if $f$ is defined on some interval $(c, +\infty)$ (resp. $(-\infty, c)$) and, for every sequence $(x_n)$ in that interval,

  $x_n \to +\infty$ (resp. $x_n \to -\infty$) implies $f(x_n) \to L$.

- For $L = \pm\infty$, similar ($f(x)$ diverges to $\pm\infty$).

**Thm** ($\varepsilon$-$\delta$ definition) : $\lim_{x \to +\infty} f(x) = L$ if

$\forall \varepsilon > 0 \ \exists M$ such that $x > M$ implies $|f(x) - L| < \varepsilon$.

Similar for other cases ($-\infty$, $L = \infty$).

**Remark** Analogs of limit theorems hold for one-sided limits and for limits at infinity.

**Examples** (a) $\boxed{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e.}$

We know that $\lim \left(1 + \frac{1}{n}\right)^n = e$.

Convert $x \in \mathbb{R}$ into $n \in \mathbb{N}$ by $[x]$. Hence $\lim_{x \to \infty} \left(1 + \frac{1}{[x]}\right)^{[x]} = \lim \left(1 + \frac{1}{n}\right)^n = e$ (why?)

$x - 1 \le [x] \le x \quad \Rightarrow$

$$\left(1 + \frac{1}{[x]+1}\right)^{[x]} \le \left(1 + \frac{1}{x}\right)^x \le \left(1 + \frac{1}{[x]}\right)^{[x]+1}$$

$$\downarrow \qquad \longleftarrow \text{why?} \longrightarrow \qquad \downarrow$$

$$e \qquad\qquad\qquad\qquad e$$

By Squeeze Thm, $\left(1 + \frac{1}{x}\right)^x \to e$ as $x \to \infty$. QED.

(b) $$\boxed{\lim_{x \to \infty} \frac{x^p}{a^x} = \begin{cases} 0, & a > 1 \\ +\infty, & 0 < a < 1 \end{cases}}$$

(Exponential vs. polynomial functions)
Compare with $\lim \frac{n^p}{a^n}$ = same (Ex. 9.14)

Proof - similar to (a); for $a > 1$ note that

$$\frac{[x]^p}{a^{[x]+1}} < \frac{x^p}{a^x} < \frac{([x]+1)^p}{a^{[x]}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$0 < a < 1$ similar (use Comparison Thm).

**Exercise** $\lim_{x\to 0^+} f(x) = \lim_{z\to+\infty} f(1/z)$ ; same for $z = -\infty$.

~~Exercise~~

**Examples** (c) $\boxed{\lim_{x\to 0}(1+x)^{1/x} = e}$

$$\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{z\to+\infty}\left(1+\frac{1}{z}\right)^z = e \text{ by Ex. (a).}$$

To complete the proof, it suffices to show that $\lim_{x\to 0^-}(1+x)^{1/x} = e$.

(the conclusion would follow from Prop. p. 63).

$$\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} = \lim_{z\to-\infty}\left(1+\frac{1}{z}\right)^z = \lim_{n\to\infty}\bigg(\underbrace{1-\frac{1}{n}}_{=\frac{n-1}{n}}\bigg)^{-n}$$ (by an argument similar to Ex.(a)) – give it!

$$= \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{\downarrow\, e}\cdot\underbrace{\left(1+\frac{1}{n-1}\right)}_{\downarrow\, 1} = e.$$

QED.

## Asymptotic Analysis (see Wikipedia)

Analysis of limiting behavior of a function $f(x)$ near a point $x_0$ or at $\infty$.

**Def** $\boxed{f(x) \approx g(x)}$ as $x\to x_0$ if $\lim_{x\to x_0}\frac{f(x)}{g(x)} = 1$

Landau's notation: $\boxed{f(x) = o(g(x))}$ as $x\to x_0$ if $\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0$

← here $x_0 \in \mathbb{R}\cup\{\pm\infty\}$.

**Examples**: (a) $\sin(x) \approx x$ as $x\to 0$ because $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\sin(x)$: complicated function; $x$: simple function

(b) $x^5 = o(x^6)$ as $x\to\infty$, $x^6 = o(x^5)$ as $x\to 0$. (WHY?)

(c) $x^p = o(a^x)$ as $x\to\infty$ for $a>1$

$a^x = o(x^p)$ as $x\to\infty$ for $0<a<1$

by Ex. (6) p. 64.

~~(d) $3x^2-5x+7 \approx 3x^2$ as $x\to\infty$ (PROVE!)~~

(d) $f(x) = 3x^2 - 5x + 2\sin\left(\frac{1}{x}\right) \approx 3x^2$ as $x\to\infty$,

$\approx 2\sin\left(\frac{1}{x}\right)$ as $x\to 0$.