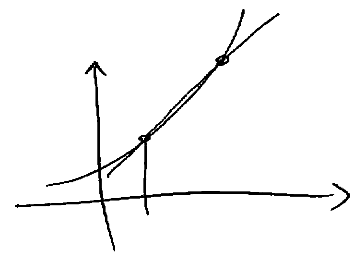
11/11/2011

§28. Differentiation

Def: f defined on an open interval $\ni a$.
f is differentiable at a if

$$f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{exists and is finite.}$$

$\uparrow$ derivative of f at a



~~Ex: f(x) =~~

Remarks 1) Rates of change $\frac{\Delta y}{\Delta x}$, velocity.
2) Tangent lines.
3) f' as function.

4) $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Examples

(a) $f(x) = x^n$ $\quad f'(a) = \frac{x^n - a^n}{x-a} = na^{n-1}$ (by HW 9)

$f'(x) = nx^{n-1}$. In particular, $\quad$ ~~(n=0)~~ $\quad \left((\text{const})' = 0,\ x' = 1\right)$

(b) $f(x) = \sin x$

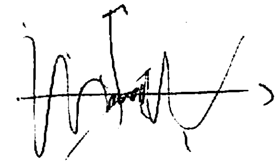
$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h} = \lim_{h\to 0} \underbrace{\frac{\sin\left(\frac{h}{2}\right)}{h/2}}_{\to 1} \cdot \cos\left(x+\frac{h}{2}\right) = \cos x.$$

$\therefore$ $(\sin x)' = \cos x$.

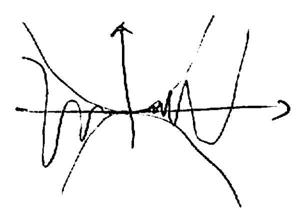
(c) $(\cos x)' = -\sin x$.

(d) $f(x) = x\sin\frac{1}{x}$. $\quad f'(0) = \lim_{x\to 0} \frac{x\sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x}$

DOES NOT EXIST



(e) $f(x) = x^2 \sin\frac{1}{x}$. $\quad f'(0) = \lim_{x\to 0} x\sin\frac{1}{x} = 0$.



(f) $f(x) = \ln x$,

$$f'(1) = \boxed{\lim_{x\to 0} \frac{\ln(1+h)}{h} = 1}$$

Proof: $= \lim_{x\to 0} \ln\left[\underbrace{(1+h)^{\frac{1}{h}}}_{\to e}\right] = \ln e = 1$. (by continuity of $\ln x$)

(g) $f(x) = e^x$. $\quad f'(x) = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right) \underset{\substack{h=\ln(1+t) \\ (\text{composition})}}{=} e^x \cdot \underbrace{\lim_{t\to 0} \frac{t}{\ln(1+t)}}_{\overset{\shortparallel}{1} \text{ by (f)}} = e^x$.

$\therefore$ $(e^x)' = e^x$.

(h) $f(x)=|x|$. $f'(x)=\{\operatorname{sign}(x), x\neq 0$; $f'(0)$ does not exist. (Although continuous).

Remark: In our argument in (f), we accepted that $\ln x$ is defined for $x>0$ and is continuous. This is not trivial.

A rigorous development would

1) Define e^x, $x\in\mathbb{N}$
2) Extend to $x\in\mathbb{Z}$, then $x\in\mathbb{Q}$, then define for $x\in\mathbb{R}$
3) Prove the basic properties of e^x, $x\in\mathbb{R}$, including continuity,
4) Define $\ln x$ as the inverse function of e^x $\Rightarrow$ continuous, too.

Alternative definitions of e^x – see Wikipedia "Characterizations of exponential function".

$\boxed{\text{Thm (28.2) If } f \text{ is differentiable at } a \text{ then } f \text{ is continuous at } a.}$

Remark (the converse is not true – see Ex. (h)).

Proof of Thm: Recall that $f'(a)=\lim\limits_{x\to a}\underbrace{\dfrac{f(x)-f(a)}{x-a}}_{=:D(x,a)}$.

$$\Rightarrow f(x)=f(a)+D(x,a)(x-a).$$

As $x\to a$, $f(x)\to f(a)+f'(a)\cdot\underbrace{\lim(x-a)}_{=0}=f(a)$. QED.

Thm (28.3) (Differentiation rules)

Let f,g be differentiable at a.
Then kf, $f+g$, $f\cdot g$, f/g (if $g(a)\neq 0$) are differentiable at a, and

(i) $(kf)'(a)=kf'(a)$
(ii) $(f+g)'(a)=f'(a)+g'(a)$
(iii) $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$
(iv) $(f/g)'(a)=\dfrac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$ if $g(a)\neq 0$.

Proof (i) – Ex

(ii) $(f+g)'(a)=\lim\limits_{x\to a}\dfrac{(f(x)+g(x))-(f(a)+g(a))}{x-a}=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}+\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}=f'(a)+g'(a)$

(iii) $(fg)'(a)=\lim\limits_{x\to a}\dfrac{f(x)g(x)-f(a)g(a)}{x-a}\underset{\substack{\text{Add, subtract}\\ f(x)g(a)}}{=}\lim\limits_{x\to a}\dfrac{f(x)[g(x)-g(a)]+g(a)[f(x)-f(a)]}{x-a}=f(a)\cdot g'(a)+g(a)\cdot f'(a)$

Because $f(x)\to f(a)$, $g(x)\to g(a)$ by continuity (Thm. 28.2).