(h) $f(x)=|x|$. $f'(x)=\{\operatorname{sign}(x), x\neq 0$; $f'(0)$ does not exist. (Although continuous)

Remark: In our argument in (f), we accepted that $\ln x$ is defined for $x>0$ and is continuous. This is not trivial.

A rigorous development would

1) Define e^x, $x\in\mathbb{N}$
2) Extend to $x\in\mathbb{Z}$, then $x\in\mathbb{Q}$, then define for $x\in\mathbb{R}$
3) Prove the basic properties of e^x, $x\in\mathbb{R}$, including continuity,
4) Define $\ln x$ as the inverse function of e^x $\Rightarrow$ continuous, too.

Alternative definitions of e^x – see Wikipedia "Characterizations of exponential function"

→ Next time, consider giving Thm 4.2 [Wade] here; then deduce Thm from it. Moreover, consider ~~introducing~~ writing $f(x+h)=f(x)+f'(x)h+o(h)$ as in [Zorich]

Thm (28.2) If f is differentiable at a then f is continuous at a.

Remark (the converse is not true – see Ex. (h)).

Proof of Thm Recall that $f'(a)=\lim_{x\to a}\underbrace{\frac{f(x)-f(a)}{x-a}}_{=:D(x,a)}$

$\Rightarrow f(x)=f(a)+D(x,a)(x-a)$.

As $x\to a$, $f(x)\to f(a)+f'(a)\cdot\underbrace{\lim(x-a)}_{=0}=f(a)$. QED

11/14/2011

Thm (28.3) (Differentiation rules)

Let f,g be differentiable at a.
Then kf, $f+g$, $f\cdot g$, f/g (if $g(a)\neq 0$) are differentiable at a, and

(i) $(kf)'(a)=kf'(a)$
(ii) $(f+g)'(a)=f'(a)+g'(a)$
(iii) $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$
(iv) $(f/g)'(a)=\dfrac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$ if $g(a)\neq 0$

Proof (i) – Ex

(ii) $(f+g)'(a)=\lim_{x\to a}\dfrac{(f(x)+g(x))-(f(a)+g(a))}{x-a}=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}+\lim_{x\to a}\dfrac{g(x)-g(a)}{x-a}=f'(a)+g'(a)$

(iii) $(fg)'(a)=\lim_{x\to a}\dfrac{f(x)g(x)-f(a)g(a)}{x-a}\underset{\substack{\text{Add, subtract}\\ f(x)g(a)}}{=}\lim_{x\to a}\dfrac{f(x)[g(x)-g(a)]+g(a)[f(x)-f(a)]}{x-a}=f(a)\cdot g'(a)+g(a)\cdot f'(a)$

Because $f(x)\to f(a)$, $g(x)\to g(a)$ by continuity (Thm. 28.2).

}

(iv) Since $g(a) \neq 0$, g continuous $\Rightarrow$ $g(x) \neq 0$ for $x \in$ some open interval $\ni a$.

$$(f/g)'(a) = \lim_{x \to a} \frac{f(x)/g(x) - f(a)/g(a)}{x-a} \;=\; ?$$

$$\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} = \frac{f(x)g(a) - g(x)f(a)}{g(x)g(a)} \underset{\substack{\text{Add, subtract} \\ f(a)g(a)}}{=\!=\!=} \frac{g(a)[f(x)-f(a)] - f(a)[g(x)-g(a)]}{g(x)g(a)}$$

$$= \lim_{x\to a} \frac{g(a)\cdot \frac{f(x)-f(a)}{x-a} - f(a)\cdot\frac{g(x)-g(a)}{x-a}}{g(x)g(a)} = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}$$

(by continuity of f, g and by def of derivative)

QED

~~Derivat~~ Differentiation in Landau's notation (See Wikipedia)

Recall: Given two functions f, g, we say

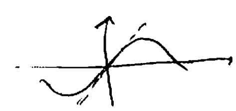
$f(x) = o(g(x))$ as $x \to x_0$ ("little o-notation") (*)

if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

- Further, the notation $o(g(x))$ sometimes is used by itself, without a reference to (*). There $o(g(x))$ denotes an unspecified function $f(x)$ such that $f(x) = o(g(x))$.

Example: (a) $\boxed{\sin x = x + o(x)}$ as $x \to 0$

"ERROR TERM"



Indeed, $\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \frac{\sin x}{x} = 1 + h(x)$ where $\lim_{x \to 0} h(x) = 1$, so $h(x) = o(1)$

$\sin x = x + x\,h(x)$; here $\lim_{x\to 0} x\,h(x) = 0$, so $x\,h(x) = o(x)$.

(b) $2 - x^2 + 3x^3 - 5(\sin x)^4 = 2 + o(x)$ as $x \to 0$

Also, $= 2 - x^2 + o(x^2)$

Also, $= 2 - x^2 + 3x^3 + o(x^3)$

Note: the unspecified function $f(x)$ in Landau's notation may change from line to line (Ex. b)

Hence: $\boxed{o(f(x)) + o(f(x)) = o(f(x)),\ o(o(f(x))) = o(f(x))}$

Let's write down the def. of derivative in Landau's notation.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

$$\Rightarrow \quad f'(x) = \frac{f(x+h)-f(x)}{h} + o(1).$$

$$\Rightarrow \quad \underbrace{f(x+h) = f(x) + f'(x)h}_{\uparrow \text{ Linearization.}} + \underbrace{h\,o(1)}_{= o(h)}.$$

$\Rightarrow$ **Proposition** (Derivative in Landau's notation).

If f is differentiable at x then

$$\boxed{f(x+h) = f(x) + f'(x)h + o(h)} \quad \text{as } h\to 0. \qquad (*)$$

Conversely, suppose that for some $F(x)$,

$$f(x+h) = f(x) + F(x)h + o(h) \quad \text{as } h\to 0.$$

Then f is differentiable at x and $f'(x) = F(x)$.

$\$$ (($\Rightarrow$) was proved; ($\Leftarrow$) - Exercise).

Equivalent form of $(*)$:

$$\boxed{f(x) = f(a) + f'(a)(x-a) + o(x-a) \quad \text{as } x\to a} \qquad (**)$$

Thm (Chain Rule) If f is diff. at a,
g is diff. at $f(a)$
then $g\circ f$ is diff. at a, and $\boxed{(g\circ f)'(a) = g'(f(a))\cdot f'(a)}$

~~**Proof.** We compute the increment~~

$$(g\circ f)(x+h) - (g\circ f)(x) = g(\underbrace{f(x+h)}_{= y+t}) - g(\underbrace{f(x)}_{= y})$$

$$\overset{\text{by }(*)}{\underset{\text{for } g}{=}} g'(y)t + o(t) = g'(f(x))[f(x+h)-f(x)] + o(f(x+h)-f(x))$$

(as $f(x+h)-f(x)\to 0$ which is true by continuity of f (as $h\to 0$))

$$\overset{\text{by }(*)}{\underset{\text{for } f}{=}} g'(f(x))[f'(x)h + o(h)] + o(f(x+h)-f(x))$$

$$= g'(f(x))f'(x)h +$$

Proof.

$$\underline{(g\circ f)(x+h)-(g\circ f)(x)} = g(\underbrace{f(x+h)}_{=y}) - g(\underbrace{f(x)}_{=a})$$

; let us apply (**) for g

$$= g'(a)(y-a) + o(y-a)$$

$$= g'(f(x))\left[f(x+h)-f(x)\right] + o\left[f(x+h)-f(x)\right]$$

(as $\overset{y}{f(x+h)} \to \overset{a}{f(x)}$ which is true by continuity of f)

Now let us apply (*) for f:

$$= g'(f(x))\left[f'(x)h + o(h)\right] + o\left[f'(x)h + o(h)\right] \qquad (\text{as } h\to 0)$$

$$= g'(f(x))f'(x)h + \underbrace{g'(f(x))\cdot o(h) + f'(x)\,o(h) + o(h)}_{=o(h)}$$

$$= \underline{g'(f(x))f'(x)h + o(h)}$$

Therefore by Prop. p.69, $(g\circ f)'(x) = g'(f(x))f'(x)$. QED.

Examples (a) $\boxed{(a^x)' = a^x\ln a}$

Proof: $(a^x)' = (e^{x\ln a})' \underset{\substack{f(x)=x\ln a\\ g(y)=e^y}}{=} e^{x\ln a}\cdot \ln a = a^x\ln a.$ QED.

(b). $\boxed{(\cos x)' = -\sin x}$

$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)' \underset{\substack{f(x)=\frac{\pi}{2}-x\\ g(x)=\sin x}}{=} \cos\left(\frac{\pi}{2}-x\right)\cdot(-1) = -\sin x.$ QED.

(c) $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \boxed{\frac{1}{\cos^2 x}}$ QED

(d) $\left(\frac{1}{f(x)}\right)' = -\frac{1}{f(x)^2}\cdot f'(x) = -\frac{f'(x)}{f(x)^2}.$