§29. The Mean Value Theorem

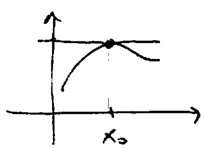
11/16/2011

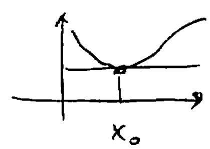
Solution of optimization problems relies on: (Eg. ~~Optimize~~ "Maximize $x - x^3$ on $[0,1]$" → $1-2x^2=0$, $x = 1/\sqrt{2}$)

Lemma (Thm 29.1) [Fermat]. ("Criterion of Local extremum")

Let f be ~~differentiable at x_0~~ defined on an open interval $\ni x_0$, and differentiable at x_0.
If f assumes its maximum or minimum at x_0 then

$$f'(x_0) = 0.$$

max: 

min: 

Proof: Suppose f is defined on (x_0-a, x_0+a) assumes max at x_0 ⇒

$$f(x_0+h) \leq f(x) \quad \text{for all } |h|<a.$$

$$\Rightarrow \quad \frac{f(x_0+h)-f(x)}{h} \leq 0 \quad \text{for } h>0,$$

$$\cancel{\frac{f(x_0+h)-f(x)}{h} \geq 0 \text{ for } h<0}.$$

$$\Rightarrow \quad \lim_{h\to 0^+} \frac{f(x_0+h)-f(x)}{h} \leq 0 \quad \text{(by Comparison Thm)}.$$

Similarly, $\displaystyle \lim_{h\to 0^-} \frac{f(x_0+h)-f(x)}{h} \geq 0.$

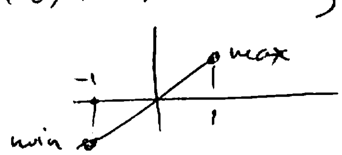
But both limits must $= \displaystyle \lim_{h\to 0} \frac{f(x_0+h)-f(x)}{h} = f'(x_0)$ (by Thm 20.10)

$\Rightarrow f'(x_0) = 0.$ QED.

Exercise: give a similar proof for the case where f attains its minimum at x_0.

Remarks:
1) Physical interpretation: velocity $=0$ at the point of return.
2) Important that ~~x_0 is an interior~~ extremum is attained at an interior point x_0 of the interval (because the interval is open).

There are many functions that attain max/min at an endpoint x_0 of $[a,b]$ but $f'(x_0) \neq 0$; e.g. $f(x) = x$ on $[-1,1]$



Existence of x_0: ?

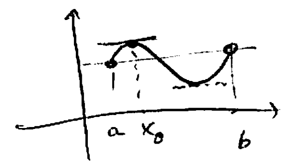
Rolle's Theorem (29.2)

Let f be continuous function $[a,b]$ and differentiable on (a,b), ~~and which satisfies~~

Suppose $f(a) = f(b)$.

Then there exists $x_0 \in (a,b)$ such that

$$f'(x_0) = 0.$$



Proof. By Extreme Value Thm (18.1), $\exists$ points $x_{min}, x_{max} \in [a,b]$ where f attains min, max,

- If $f(x_{min}) = f(x_{max})$ then $f = \text{const} \Rightarrow f'(x)$ for all $x \in (a,b)$.
- If $f(x_{min}) < f(x_{max})$, then since $f(a) = f(b)$, one of the points x_{min}, x_{max} must lie in (a,b). ~~By Fermat's lemma, the derivative~~

  Denote it by x_0. By Fermat's lemma, $f'(x_0) = 0$. QED

Mean Value Theorem

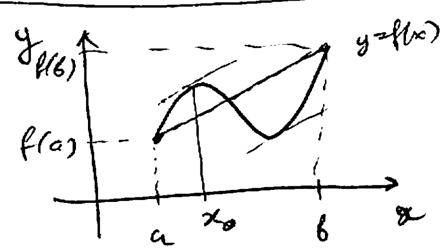
Let f be continuous on $[a,b]$, differentiable in (a,b).

Then there exists $x_0 \in (a,b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Proof Consider the linear function $L(x)$ whose graph connects $(a, f(a))$, $(b, f(b))$.

Then $L'(x) = \dfrac{f(b) - f(a)}{b - a}$,



Apply Rolle's Theorem for $g(x) = f(x) - L(x)$.

$g(a) = g(b) = 0$, $\Rightarrow \exists\ x_0 \in (a,b)$: $g'(x_0) = f'(x_0) - L'(x_0) = 0$.

QED

Remarks 1) Physical interpretation: ~~instantaneous speed~~

instantaneous ~~velocity~~ speed = average speed at some point in time.

~~2) f'(x) is local quant~~

2) MVT connects local and global behavior of f.

Consequences of MVT

① Derivative in Landau's notation.

Recall: $\forall x$: $f(x+h) = f(x) + f'(x)h + o(h)$ as $h \to 0$.

~~MVT $\Rightarrow \forall x$ $f(x+h) = f(x) +$~~

MVT $\Rightarrow$ for every h (small enough so that f is differentiable on $(x, x+h)$):

$$\boxed{f(x+h) = f(x) + f'(x_0)h \quad \text{for some } x \le x_0 \le x+h}$$

(subtract $f(x)$ from both sides & divide by h to see this)

② Antiderivative

Cor If $f'(x) = 0$ for all $x \in (a,b)$ then $f = \text{const}$ on (a,b).

Proof If f were not const $\Rightarrow$ there exist $a < a' < b' < b$ s.t. $f(a') \neq f(b')$.

MVT $\Rightarrow$ there exists $x_0 \in (a', b')$ s.t. $f'(x_0) = \dfrac{f(b') - f(a')}{b' - a'} \neq 0$. ↯ QED

Cor If $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists const C s.t.
$$f(x) = g(x) + C \quad \text{for } x \in (a,b)$$

Proof Apply previous Cor for the function $f(x) - g(x)$. QED.

Remark: Cor states that the antiderivative F of f ("indefinite integral": $F' = f$) is defined uniquely up to additive const:

$$\int \underbrace{\cos x}_{f(x)} \, dx = \underbrace{\sin x}_{F(x)} + C$$