~~More cons~~ In the last class, we analyzed when $f'(x)=0$. (extrema).

Another consequence of Mean Value Theorem:

Cor (29.7) (Monotonic functions). • If $f'(x) \geq 0$ for all ~~x∈(a,b)~~ points in some interval then f is non-decreasing on ~~(a,b)~~ that interval

(and if $f'(x) > 0$ - - - - then f is increasing - - - .)

• Similarly, if $f'(x) \leq 0$ for all x ~~∈(a,b)~~ then f is non-increasing ~~on (a,b)~~,

(and if $f'(x) < 0$ then f is decreasing . . .)

Proof ~~Let $a<x_1<x_2<b$. Then by M.V.T.,~~ Let $x_1 < x_2$ be points in the interval.

By M.V.T., $f(x_2) - f(x_1) = f'(x_0)\underbrace{(x_2 - x_1)}_{> 0}$ for some $x_0 \in (x_1, x_2)$

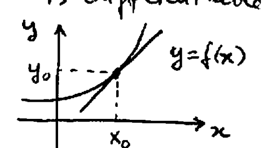
So, if $f'(x_0) \geq 0$ then $f(x_2) - f(x_1) \geq 0$, hence non-decreasing.

Other parts are similar. QED.

Example: $f(x) = x^2$. $f'(x) = x$, so increasing on $(0, +\infty)$, decreasing on $(-\infty, 0)$. .

Thm (≈ 29.9) (Derivative of inverse function). Suppose f is one-to-one ~~and continuous~~ on some open interval containing x_0 and both f and f^{-1} are continuous at x_0 and $y_0 = f(x_0)$ respectively. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \qquad (*)$$



~~Proof First note that the increments $f(x)-f(x_0)$ and $f^{-1}(y)-f^{-1}(y_0)$ are never zero when $x \neq x_0$. Now, we want to prove the existence~~ ~~let $y=f(x)$.~~

Remark If we knew that f^{-1} is differentiable at y_0 then $(*)$ is easy:

~~by composition~~ $(f^{-1} \circ f)(x) = x$

$\Rightarrow$ differentiating this composition at x_0 ~~yields~~ by chain rule yields

$(f^{-1})'(y_0) \cdot f'(x_0) = 1 \quad \Rightarrow \ (*)$ follows.

Proof Let $y=f(x)$. Note that the increments $f(x)-f(x_0)$ and $f^{-1}(y)-f^{-1}(y_0)$ are not zero if $x\neq x_0$. We want to show the existence of

$$(f^{-1})'(y_0)=\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}. \qquad (**)$$

Make change of variable $y=f(x)$. By continuity,

$y\to y_0$ implies (and is in fact equivalent to) $\underset{f^{-1}(y)}{\overset{\|}{x}}\to\underset{f^{-1}(y_0)}{\overset{\|}{x_0}}$

So

$$(**)=\lim_{x\to x_0}\frac{x-x_0}{f(x)-f(x_0)}=\frac{1}{f'(x_0)}. \quad \text{(by def. of } f'(x_0) \text{ and limit of ratio)}$$

QED.

Exercise: Justify the change of variable in this argument

$\left(\forall y_n\to y_0 \Rightarrow x_n:=f^{-1}(y_n)\to f^{-1}(y_0)=x_0. \text{ So by }, \lim\frac{x_n-x_0}{f(x_n)-f(x_0)}=\frac{1}{f'(x_0)}\right.$

$\left.\qquad \| \qquad \lim\frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}\Rightarrow (**)=\frac{1}{f'(x_0)}\right)$

Examples

(a) Logarithmic function: $y=f(x)=e^x$, $x=f^{-1}(y)=\ln y$.

$$(f^{-1})'(y)=\frac{1}{f'(x)}=\frac{1}{e^x}=\frac{1}{y}. \quad\Rightarrow\quad \boxed{(\ln x)'=\frac{1}{x}}$$

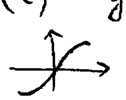
(b) ~~Polynomial~~ Power (polynomial) function $f(x)=x^p$, $p\in\mathbb{R}$

$$(x^p)'=(e^{p\ln x})'=e^{p\ln x}\cdot\frac{p}{x} \qquad \text{(by Chain rule)}$$

$$=x^p\cdot\frac{p}{x}=px^{p-1}.$$

$$\boxed{(x^p)'=px^{p-1},\ p\in\mathbb{R}}$$

(c) $y=f(x)=\sin x$. $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $y\in[-1,1]$ $\quad x=f^{-1}(y)=\sin^{-1}y$. $\quad (f^{-1})'(y)=\frac{1}{f'(x)}=\frac{1}{\cos x}=\frac{1}{\sqrt{1-\sin^2 x}}=\frac{1}{\sqrt{1-y^2}}$.



$$\boxed{(\sin^{-1}x)'=\frac{1}{\sqrt{1-x^2}},\quad x\in(-1,1)}$$

§30 L'Hopital's ~~Theorem~~ Rule

Thm 30.2 (Partial case of L'Hopital's Rule). Let f, g be differentiable ~~on~~ &

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = +\infty, \quad \lim_{x\to+\infty} g(x) \to +\infty$$

Then $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = +\infty$$

Remark: In general, $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$.

Lemma 30.1 (Generalized MVT) [Cauchy] Let f, g be continuous on $[a,b]$, diff. on (a,b). There exists $x_0 \in (a,b)$ s.t.

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

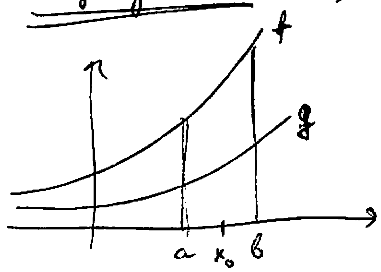
Proof $h(x) := f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$.

Want to show: $h'(x_0)=0$ for some $x_0 \in (a,b)$

Since $h(a)=h(b)$, Rolle's theorem gives us such x_0. QED.

Proof of Thm. Let $M>0$. By assumption, $\exists\, a$ such that

$$\frac{f'(x)}{g'(x)} > M \quad \text{for } x>a.$$



By Generalized MVT, $\forall\, b>a \ \exists\, x_0 \in (a,b)$ such that

$$R := \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)} > M.$$

Now we get rid of $f(a), g(a)$ in the left hand side. Keep a fixed, let $b\to\infty$:

$$R = \frac{\frac{f(b)}{g(b)} - \frac{f(a)}{g(b)}}{1 - \frac{g(a)}{g(b)}} > M \qquad \frac{f(a)}{g(b)} \to 0, \quad \frac{g(a)}{g(b)} \to 0$$

Choose b_0 so that these two fractions are smaller than $\frac{1}{2}$ in absolute value for $b>b_0$.

$$\Rightarrow \frac{\frac{f(b)}{g(b)} + \frac{1}{2}}{1-\frac{1}{2}} > M \Rightarrow \frac{f(b)}{g(b)} > \frac{M-1}{2} \text{ for } b>b_0$$

Summarizing: $\forall M\ \exists\, b_0$ s.t. $\frac{f(b)}{g(b)} > \frac{M-1}{2}$ for $b>b_0$. $\Rightarrow \lim_{b\to\infty} \frac{f(b)}{g(b)} = +\infty$. QED

Example (logarithmic vs polynomial function)

$$\boxed{\lim_{x\to+\infty} \frac{x^p}{\ln x} = +\infty \quad \text{for } p>0.}$$

Indeed, $\displaystyle\lim_{x\to+\infty} \frac{(x^p)'}{(\ln x)'} = \lim_{x\to+\infty} \frac{p x^{p-1}}{1/x} = p \lim_{x\to+\infty} x^p = +\infty$. L'HOPITAL $\Rightarrow$ QED.

Hence, hierarchy:

$$\boxed{\ln x \ll x^p \ll e^x} \quad \text{as } x\to\infty \text{ for all } p>0.$$