11/21/2011

§31. Taylor's Theorem

Remainder of linearization

- Let f be differentiable at x_0.

Then we can write down the linearization of f around x_0 as follows:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0) \qquad \text{as } x \to x_0$$

(see (**) p. 69).

- We will be interested in the form of the remainder $o(x-x_0)$. (formula? estimate?)

Second derivative f'' will play a role;

~~So let us write~~

$$\cancel{f(x) = L(x_0;x) + r(x_0;x)} \qquad (*)$$

~~where $L(x_0;x) = f(x_0) + f'(x_0)(x-x_0)$ is linear in x,~~

~~$r(x_0;x)$ is a remainder defined by (*).~~

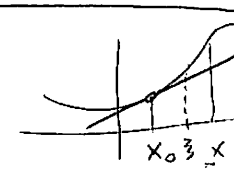
Theorem (Taylor's Theorem for $n=1$)

Let f be continuously differentiable on an interval with endpoints x_0, x.

Then there exists ξ between x_0, x such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2.$$

Proof Let us write $f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{= L(x_0;x)} + r(x_0;x)$



Here $L(x_0;x)$ is a linear function, and ~~$r(x_0;x)$ that we need to estimate.~~

$r(x_0;x) = f(x) - L(x_0;x)$ is the remainder that we need to estimate.

Idea: try writing down linearization around a point ξ different from x_0; ~~estimate~~ ~~$\xi \in (x_0, x)$~~

$$\cancel{r(\xi;x) = f(x) - L(\xi;x) = f(x) - [f(\xi) + f'(\xi_0) +}$$

namely for some point ξ between x_0, x.

Check the remainder

$$r(\xi; x) = f(x) - \left[f(\xi) + f'(\xi)(x-\xi) \right] =: F(\xi);$$

use MVT for $F(\xi)$ on $[x_0, x]$

$F(x_0) = r(x_0; x)$ ← this is what we need to estimate

$F(x) = 0$.

$F'(\xi) = -\left[f'(\xi) + f''(\xi)(x-\xi) \right] = -f''(\xi)(x-\xi)$. (by product rule)

MVT $\Rightarrow$ $\exists\ \xi$ between x_0 and x such that

$$F'(\xi) = \frac{F(x) - F(x_0)}{x - x_0}$$

$$\Leftrightarrow \quad -f''(\xi)(x-\xi) = \frac{-r(x_0; x)}{x - x_0}$$

$$\Leftrightarrow \quad \boxed{r(x_0; x) = f''(\xi)(x-\xi)(x-x_0)} \qquad (*)$$

~~This is good (quadratic!) but not quite what we claimed, (*) is called the Cauchy form of the remainder.~~

This is good (quadratic in $x - x_0$). (*) is called the Cauchy form of the remainder.
But (*) is not quite what we claimed.

- Let's apply Generalized MVT instead, with $G(\xi) = (x-\xi)^2$.

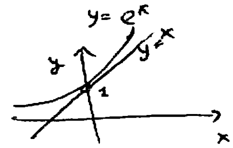
$$\frac{F'(\xi)}{G'(\xi)} = \frac{\overset{0}{F(x)} - F(x_0)}{\underset{0}{G(x)} - G(x_0)}$$

$$\frac{-f''(\xi)(x-\xi)}{2(x-\xi)} = \frac{-r(x_0; x)}{(x-x_0)^2}$$

$$(**) \qquad \boxed{r(x_0; x) = \frac{f''(\xi)}{2}(x-x_0)^2}, \qquad \text{as claimed.}$$

QED

Remark: (**) is called the Lagrange form of the remainder.



Example (a) $\boxed{e^x \geq 1+x \quad \text{for all } x \in \mathbb{R}.}$

~~By~~ Use Theorem p. 77, for $x_0 = 0$, $f(x)=e^x$, $\Rightarrow$ For every $x \in \mathbb{R}$ there exists ξ:

$$e^x = e^0 + (e^x)'(0)\cdot x + \frac{(e^x)''(\xi)}{2}x^2$$

$$e^x = 1 + x + \underbrace{\frac{e^\xi}{2}x^2}_{\geq 0}.$$

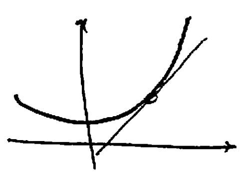
QED.

~~A similar argument works for an arbitrary function: with $f''(\xi)>0$ (convex).~~
~~$f(x) \geq f(x_0) + f'(x_0)(x-x_0)$ if $f''(x)>0$~~

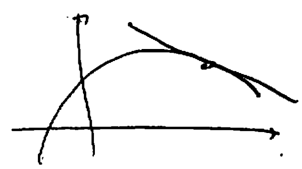
Similar argument $\Rightarrow$

Cor (Convexity). Suppose $f''(x) > 0$ for all $x \in [a,b]$ ("Convex")
Then $f(x) \geq f(x_0) + f'(x_0)(x-x_0)$ for $x \in [a,b]$.

Similarly, if $f''(x) < 0$ for all $x \in [a,b)$ ("Concave")
then $f(x) \leq f(x_0) + f'(x_0)(x-x_0)$.



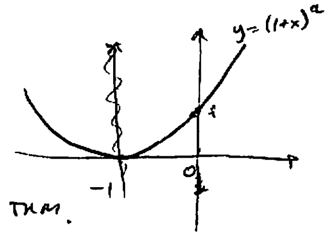
convex



Concave.

Example (b) [Bernoulli inequality]

$$\boxed{(1+x)^a \geq 1 + ax \quad \text{for } a \geq 1,\ x \geq 0.}$$



RECALL: FOR $a \in \mathbb{N}$ THIS IS A CONSEQUENCE OF BINOMIAL THM.

Use Thm p. 77 for $f(x) = (1+x)^a$, $x_0 = 0$.

$$f'(x) = a(1+x)^{a-1}, \quad f''(x) = a(a-1)(1+x)^{a-2}$$

$$(1+x)^a = 1 + ax + \underbrace{\frac{a(a-1)}{2}(1+\xi)^{a-2}x^2}_{\geq 0} \geq 1+ax. \quad (\exists\ \xi \text{ between } 0, x).$$

QED.

$\uparrow$ Problem: <0 if ξ is a very negative number.

~~Use Cauchy form of the remainder instead (p.78).~~

~~$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}(1+\xi)^{a-2}(x-\xi)x$~~ ~~(wrong!)~~ ~~$\Rightarrow$ QED.~~