§ 31. Taylor's Theorem (for general n).

Approximation by polynomial functions.

So far, we have studied <u>linearization</u> of $f(x)$, i.e. approximation of f by a linear function.

Next step: by quadratic, cubic, ..., n'th degree polynomial.

- We want: $f(x) = P_n(x) \approx a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n$ around x_0

- To find coefficients a_k, we insist that all derivatives agree at x_0

$$\left.\begin{aligned} f(x_0) &= P(x_0) = a_0 \\ f'(x_0) &= P'(x_0) = a_1 \\ f''(x_0) &= P''(x_0) = 2!\,a_2 \\ &\vdots \\ f^{(n)}(x_0) &= P^{(n)}(x_0) = n!\,a_n \end{aligned}\right\} \Rightarrow \quad a_k = \frac{f^{(k)}(x_0)}{k!}$$

Def Taylor's polynomial of f around x_0 of degree n is

$$P_n(x_0;x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

Taylor's thm says that $f(x) \approx P_n(x_0;x)$ if $x \approx x_0$:

Thm (<u>Taylor's Thm</u>) Assume that f has $n+1$ continuous derivatives on an interval with endpoints x_0, x.

Then there exists ξ between x_0 and x such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{= \, r_n(x_0;x)}$$

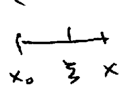
Remark 1). This form of the remainder $r_n(x_0;x)$ is called the <u>Lagrange form</u>

Another well-known form is called Cauchy: $\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-x_0)$.

One can obtain Cauchy form by using plain MVT in the proof below.

2) $\boxed{r_n(x_0;x) = o\left((x-x_0)^{n+1}\right) \text{ as } x \to x_0}$. (Peano form of the remainder)

(This is true because $\underset{\xi \to x_0}{\Rightarrow}$ $f^{(n+1)}(\xi) \to f^{(n+1)}(x_0)$ by continuity.)



Proof (similar to the proof for $n=1$, see p. 77).

We write $f(x) = P_n(x_0; x) + r(x_0; x)$

and we want to estimate the remainder $r(x_0; X)$.

Idea: $\begin{cases} \text{write down Taylor's polynomial around a point } \xi \text{ between } x_0, x \text{ (rather than around } x_0\text{);} \\ f(x) = P_n(\xi; x) + r(\xi; x). \end{cases}$

$\Rightarrow \quad r(\xi; x) = f(x) - P_n(\xi; x) =: F(\xi)$

Use MVT for $F(\xi)$ on $[x_0, x]$:

$F(x_0) = r(x_0, x)$ ← this is what we need to estimate

$F(x) = f(x) - f(x) = 0.$

$$F'(\xi) = -\frac{d}{d\xi} P_n(\xi; x) = -\Big[f'(\xi) + \underbrace{\frac{f''(\xi)}{1!}(x-\xi) - \frac{f'(\xi)}{1!}} + \underbrace{\frac{f'''(\xi)}{2!}(x-\xi)^2 - \frac{f''(\xi)}{1!}(x-\xi)} +$$

$$+ \dots + \underbrace{\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n - \frac{f^{(n)}(\xi)}{(n-1)!}(x-\xi)^{n-1}} \Big] = -\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n$$

Gen. $\underbrace{\text{MVT}}_{\text{with } G(\xi) \text{ TBD}}$ $\Rightarrow$ $\exists\, \xi$ between x_0 and x such that

$$\frac{F'(\xi)}{G'(\xi)} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)}$$

$$-\frac{f^{(n+1)}(\xi)}{n!\, G'(\xi)}(x-\xi)^n = \frac{-r(x_0; x)}{G(x) - G(x_0)}$$

$$r(x_0; x) = \frac{G(x) - G(x_0)}{G'(\xi)} \cdot \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n.$$

We choose $G(\xi) = (x-\xi)^{n+1}$. $\Rightarrow$

$$\frac{G(x) - G(x_0)}{G'(\xi)} = \frac{-(x-x_0)^{n+1}}{-(n+1)(x-\xi)^n}$$

$$\Rightarrow \quad r(x_0; x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

QED

~~Remark (Remainder is "in Lagrange form".~~
~~"Lagrange form of the remainder".~~
~~If we use plain MVT instead of generalized MVT, we would get a Cauchy form of remainder.~~

Cor. 31.4 (Taylor Series)

Assume that f is infinitely differentiable on an open interval I containing x_0, and that there exists M s.t.

$$|f^{(n)}(x)| \le M^n \qquad \text{for all } x \in I,\ n \in \mathbb{N}.$$

Then for every $x \in I$, $f(x)$ can be represented as a convergent series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k .$$

This series is called Taylor's series of f about x_0 (or, "with center x_0")

Proof

$$|r_{n-1}(x_0; x)| = \frac{|f^{(n)}(\xi)|}{n!}|x-x_0|^n \le \frac{(M|x-x_0|)^n}{n!} \to 0 \quad \text{as } n \to \infty .$$

QED

(recall that $\lim \frac{a^n}{n!} = 0$ by Ratio Test - Ex. 9.15)

Hence the series converges.

Remark We ~~say~~ showed (In Cor. 31.4) that Taylor series "converges pointwise" (i.e. for every x).

Examples (a) $f(x) = e^x$, $x_0 = 0$. $f^{(n)}(x) = 1$ for all $n \Rightarrow$

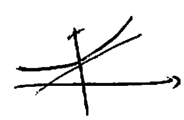
$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for every } x \in \mathbb{R}.$$

In particular, for $x=0$ we have:

$$\boxed{e = \sum_{k=0}^{\infty} \frac{1}{k!}}$$

Another consequence: Taylor series with $n=1$ in Peano form:
~~Farther, with n=1 term~~:

$$\boxed{e^x = 1 + x + o(x)} \quad \text{(linearization)}$$



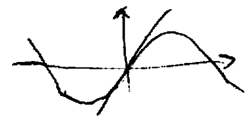
$$\hookrightarrow \boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$ (Computed already in Ex. (g) p. 66).

(b) $f(x) = \sin x$ or $f(x) = \cos x$, $x_0 = 0$: $\quad$ ($f^{(n)}(0) \in \{0, 1, -1\}$) $\Rightarrow$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad \text{for all } x \in \mathbb{R}$$

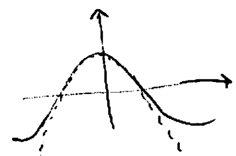
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad \text{for all } x \in \mathbb{R}.$$

In particular, $\sin x = x + o(x)$ as $x \to 0$



$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$ as we already computed in Ex. (4) p 64.

Also, $\boxed{\cos x = 1 - \frac{x^2}{2} + o(x^2)}$ as $x \to 0$.



(c) See Taylor's series of other common functions on Wikipedia.

For example,

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad \text{for } -1 \le x \le 1.$$

In particular, for $x = 1$:

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$ (Gregory & Leibnitz).

HAPPY THANKSGIVING!