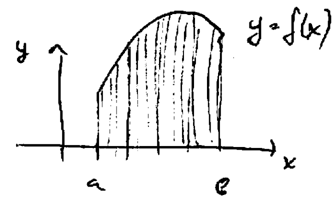
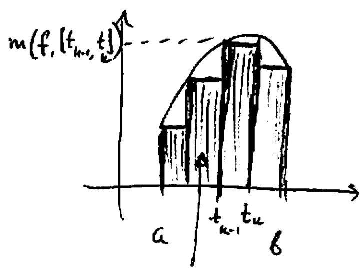
§32. The Riemann Integral

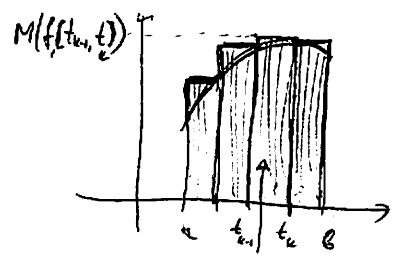
- Remark on <u>Lebesgue</u> integral.
- <u>how to define</u> R.I.?
- Recall: for $f \geq 0$, $\int_a^b f(x)\,dx = $ <u>area</u> under the graph



- But what is <u>area</u>?
- Defining area ~~as~~ by <u>partitioning</u> a shape into simpler shapes (squares, or rectangles).



<u>"Lower integral sum"</u>



<u>"Upper integral sum"</u>

- If both lower and upper integral sums $\to S$ (a number) as we refine the partition, then we would say that f is integrable on $[a,b]$ and <u>define</u> $\int_a^b f(x)\,dx := S$.

<u>Formally</u>:

<u>Def</u> (32.1) (<u>Partitions, lower/upper integral sums</u>). Let f be a bounded function on $[a,b]$.

A <u>partition</u> of $[a,b]$ is an ordered finite set $P = \{t_i\}$ such that

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

Given a <u>partition</u> P,
The <u>lower</u> and <u>upper</u> integral sums are defined as

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1}); \qquad U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1})$$

where for $S \subseteq [a,b]$ we set

$$m(f,S) = \inf\{f(x) : x \in S\}, \qquad M(f,S) = \sup\{f(x) : x \in S\}.$$

<u>Remark</u>: We need f to be <u>bounded</u> in order that $m(f,S)$, $M(f,S)$ be finite.
Unbounded functions will <u>not</u> be integrable.

Let's work with the integral sums before giving the def. of integrability.

Examples: (a) $f(x) = c = \text{const}$ $\Rightarrow$ for any partition P,

$$L(f,P) = U(f,P) = c(b-a).$$

(Thus we should say $\int_a^b c\,dx = c(b-a)$.)

~~This follows from~~ $m(f,[t_k,t_{k-1}]) = M(f,[t_k,t_{k-1}]) = c \Rightarrow$

(Indeed,
$$L(f,P) = \sum_{k=1}^{n} c(t_k - t_{k-1}) = c\sum_{k=1}^{n}(t_k - t_{k-1}) = c(t_n - t_0) = c(b-a),$$
and similarly for $U(f,P)$. QED)

(b) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$ on $[a,b]$ ("Dirichlet function")

For any partition P, $m(f,[t_k,t_{k-1}]) = 0$ and $M(f,[t_k,t_{k-1}]) = 1$

(since by the denseness of $\mathbb{Q}$ and $\mathbb{I}$, there exist rational and irrational points in $[t_k, t_{k-1}]$).

$\Rightarrow L(f,P) = 0, \quad U(f,P) = b-a \neq 0$ if $b>a$.

Thus we should say that f is <u>not integrable</u>.

~~(32.2) Prop (Properties of lower/upper integral sums)~~
~~(a) Let f be~~

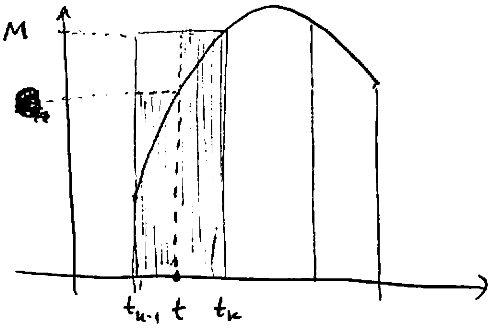
Remark Clearly, $L(f,P) \le U(f,P)$. Moreover:

> **Prop (32.2)** (<u>Refinements</u>). Let f be a bounded function on $[a,b]$, and consider partitions $P \subseteq Q$ (we say: "Q is a <u>refinement</u> of P"). Then
> $$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof The middle inequality follows from the Remark.
We shall prove $U(f,Q) \le U(f,P)$;
the other inequality is similar (see textbook)

By induction we may assume that $Q = \{t\} \cup P$;

let $t \in [t_{k-1}, t]$.



$$U(f,P) - U(f,Q) = \underbrace{M(f,[t_{k-1},t_k])}_{\overset{!!}{M}}\cdot(t_k - t_{k-1}) - \overbrace{M(f,[t_{k-1},t])}^{M \,\geq}\cdot(t - t_{k-1})$$

$$- \underbrace{M(f,[t,t_k])}_{\leq\, M}(t_k - t).$$

M (here sup is taken over a smaller set $[t_{k-1}, t] \subseteq [t_{k-1}, t_k]$)

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$$\geq M(t_k - t_{k-1}) - M(t - t_{k-1}) - M(t_k - t) = 0.$$

Q.E.D.

Cor (32.3) Let f be a bounded function on $[a,b]$, let P, Q be <u>any</u> two partitions on $[a,b]$. Then

$$L(f,P) \leq U(f,Q).$$

Proof Consider the partition $P \cup Q$; it is a refinement of both P and Q $\Rightarrow$ by Propp. 86,

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P \cup Q) \leq U(f,Q).$$

Q.E.D.

Def (32.1) (Integral) Let f be a bounded function on $[a,b]$. Define

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\},$$

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

If $L(f) = U(f)$ then f is called <u>integrable</u> on $[a,b]$, and the value of the ~~integral~~ (Riemann definite) integral is defined as

$$\int_a^b f(x)\,dx := L(f) = U(f).$$

Remark (32.4) One always has $\boxed{L(f) \leq U(f)}$ by Cor. above.

Examples (a). $f(x) = c = \text{const.}$ on $[a,b]$

In Example (a) p. 86 we showed that $L(f,P) = U(f,P) = c(b-a) \Rightarrow L(f) = U(f) = c(b-a) \Rightarrow f$ is integrable, and

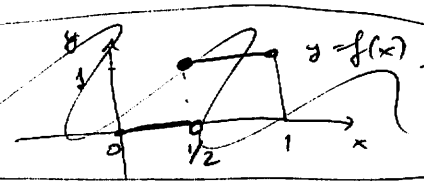
$$\boxed{\int_a^b c\,dx = c(b-a).}$$

(b) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$ on $[a,b]$

In Example (b) p. 86 we showed that $L(f,P)=0$, $U(f,P) = b-a$ for all P.

$\Rightarrow L(f) = 0$, $U(f) = b-a \Rightarrow f$ is <u>not integrable</u>.

~~(c) $f(x) = \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1 \end{cases}$~~



An equivalent def, more user friendly:

$$\boxed{\begin{array}{l}\textbf{Proposition (32.5)}\ \text{A bounded function } f \text{ on } [a,b] \text{ is integrable} \\ \quad \text{if and only if for each } \varepsilon > 0 \text{ there } \underline{\text{exists}} \text{ a partition } P \text{ of } [a,b] \text{ such that} \\ \qquad U(f,P) - L(f,P) < \varepsilon.\end{array}}$$

Proof ($\Leftarrow$: <u>Sufficiency</u>). Let $\varepsilon > 0$; choose P s.t. $U(f,P) - L(f,P) < \varepsilon$.

Recall that $U(f) \le U(f,P)$, $L(f) \ge L(f,P)$

$\Rightarrow U(f) - L(f) \le \varepsilon$.

Since this holds for all $\varepsilon > 0 \Rightarrow U(f) - L(f) = 0 \Rightarrow f$ is integrable.

($\Rightarrow$: <u>Necessity</u>). Suppose f is integrable $\Rightarrow$

$$\underbrace{U(f)}_{\inf\{U(f,P)\}} = \underbrace{L(f)}_{\sup\{L(f,P)\}}.$$

By def of inf, sup: For every $\varepsilon > 0$ there exist partitions P_1, P_2 such that

$$U(f,P_1) < U(f) + \varepsilon/2,$$
$$L(f,P_2) > L(f) - \varepsilon/2.$$

~~$\Rightarrow U(f,P) - L(f,Q) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ QE~~

Choose $P = P_1 \cup P_2 \underset{\text{by Prop. 32.2}}{\Rightarrow}$ $\left.\begin{array}{l} U(f,P) \le U(f,P_1) \le U(f) + \varepsilon/2 \\ L(f,P) \ge L(f,P_2) \ge L(f) - \varepsilon/2 \end{array}\right\} \Rightarrow U(f,P) - L(f,P) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

<u>QED</u>