§ 32. The Riemann Integral

- Remark on Lebesgue integral.

- What is R.I.? $\int f(x) \, dx = \text{area under the graph}$

- But what is area?

- Defining area by partitioning a shape into simpler shapes (squares or rectangles).

  "Lower integral sum"  
  "Upper integral sum"

- If both lower and upper integral sums $\to S$ (a number) as we refine the partition, then we would say that $f$ is integrable on $[a, b]$ and define $\int_a^b f(x) \, dx := S$.

Formally:

Def. (Partitions, Lower/Upper integral sums). Let $f$ be a function on $[a, b]$.

A partition of $[a, b]$ is an ordered finite set $P = \{t_0, t_1, \ldots, t_n\}$ such that

$a = t_0 < t_1 < t_2 < \cdots < t_n = b$.

Given a partition $P$,

The lower and upper integral sums are defined as

$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$;

$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$,

where for $S \subseteq [a, b]$ we set

$m(f, S) = \inf \{ f(x) : x \in S \}$;

$M(f, S) = \sup \{ f(x) : x \in S \}$.

Remark: We need $f$ to be bounded in order that $m(S), M(S)$ be finite.

Unbounded functions will not be integrable.
(let's work with the integral sums before giving the def. of integrability.

Examples: \( f(x) = \mathbf{1} \text{ if } x = \text{const} \Rightarrow \) for any partition \( P \),
\[ L(f, P) = U(f, P) = \mathbf{1}(b-a) \]

Thus we should say \( \int_a^b f(x) \text{d}x = (b-a) \).

This follows from
\[ M(f, \{ t_k, t_{k+1} \}) = M(f, \{ t_k, t_{k+1} \}) = \mathbf{1} \Rightarrow \]
\[ L(f, P) = \sum_{k=1}^{N} \mathbf{1}(t_k - t_{k+1}) = \mathbf{1} \sum_{k=1}^{N} (t_k - t_{k+1}) = \mathbf{1}(t_N - t_0) = \mathbf{1}(b-a) \]
and similarly for \( U(f, P) \).

QED

(6) \( f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \) (Dirichlet function).

For any partition \( P \), \( M(f, \{ t_k, t_{k+1} \}) = 0 \) and \( M(f, \{ t_k, t_{k+1} \}) = 1 \)
(since by the denseness of \( \mathbb{Q} \) and \( \mathbb{R} \), there exist rational and irrational points in \( \{ t_k, t_{k+1} \} \).

\[ \Rightarrow L(f, P) = 0, \ U(f, P) = b-a, \neq 0 \text{ if } b > a. \]

Thus we should say that \( f \) is not integrable.

\[ \int_{a}^{b} f(x) \, \text{d}x \]

Remark: Clearly, \( L(f, P) \leq U(f, P) \). Moreover:

Prop (32.3) (Refinements). Let \( f \) be a bounded function on \([a, b]\),
and consider partitions \( P \subseteq Q \) (we say: "\( Q \) is a refinement of \( P \)"). Then
\[ L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P). \]

Proof: The middle inequality follows from the Remark.
We shall prove \( U(f, Q) \leq U(f, P) \):

The other inequality is similar (see textbook).
By induction we may assume that $Q = \{ t \} \cup P$.
let \( t \in \{ t_{k-1}, t \} \).

\[
U(f, P) - U(f, Q) = \sum_{i=1}^{n} M(f, [t_{k-1}, t]) \cdot (t_k - t_{k-1}) - \sum_{i=1}^{n} M(f, [t_{k-1}, t]) \cdot (t_k - t_{k-1})
\]

\[
\geq M(t_k - t_{k-1}) - M(t_k - t_{k-1}) - M(t_k - t) = 0.
\]

**Cor (32.3)*** Let \( f \) be a bounded function on \([a,b]\),
let \( P, Q \) be any two partitions on \([a,b]\). Then
\( L(f, P) \leq U(f, Q) \).

**Proof** Consider the partition \( P \cup Q \); it is a refinement of both \( P \) and \( Q \) \( \Rightarrow \) by Prop. 36,
\( L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q) \).

**Def (32.1) (Integral)** Let \( f \) be a bounded function on \([a,b]\). Define
\[
L(f) = \sup \{ L(f, P) : P \text{ is a partition of } [a,b] \},
\]
\[
U(f) = \inf \{ U(f, P) : P \text{ is a partition of } [a,b] \}.
\]

If \( L(f) = U(f) \) then \( f \) is called integrable on \([a,b]\), and the
value of the integral (Riemann definite) integral is defined as
\[
\int_{a}^{b} f(x) \, dx = L(f) = U(f).
\]

**Remark** One always has
\[
L(f) \leq U(f) \quad \text{by Cor. above}.
\]
Examples (a). \( f(x) = c = \text{const.} \) in (4.1b)

In Example (a) p. 86 we showed that \( L(f) = U(f) = c \) \( \Rightarrow \) \( f \) is integrable, and

\[
\int_a^b c \, dx = c(b-a).
\]

(b) \( f(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x = 0 \end{cases} \) on \((0,1)\)

In Example (b) p. 86 we showed that \( L(f) = 0 \), \( U(f) = b-a \) for all \( P \).

\( \Rightarrow \) \( L(f) = 0 \), \( U(f) = b-a \) \( \Rightarrow \) \( f \) is not integrable.

(c) \( f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \)

An equivalent def, more user friendly:

Proposition (3.1.5) A bounded function \( f \) on \((a,b)\) is integrable

\( \iff \) for each \( \epsilon > 0 \) there exists a partition \( P \) of \((a,b)\) such that

\[
U(f,P) - L(f,P) < \epsilon.
\]

Proof (\( \Rightarrow \)): Sufficiency.

\( \Uparrow \) \( \epsilon > 0 \); choose \( P \) s.t. \( U(f,P) - L(f,P) < \epsilon \).

Recall that \( U(f) \leq U(f,P) \), \( L(f) \geq L(f,P) \)

\( \Rightarrow U(f) - L(f) \leq \epsilon \).

Since this holds for all \( \epsilon > 0 \) \( \Rightarrow U(f) = L(f) = 0 \) \( \Rightarrow \) \( f \) is integrable.

(\( \Leftarrow \)): Necessity.

Suppose \( f \) is integrable. \( \Rightarrow \)

\[
U(f) = L(f)
\]

\[
\inf \{ U(f,P) \} = \sup \{ L(f,P) \}.
\]

By def. of inf. \( \forall \epsilon > 0 \) there exist partitions \( P, P_2 \) such that

\[
U(f,P) < U(f) + \frac{\epsilon}{2},
\]

\[
L(f,P_2) > L(f) - \frac{\epsilon}{2}.
\]

\( \Rightarrow U(f,P) - L(f,P_2) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \) \( \Box \).

Choose \( P = P \cup P_2 \) \( \Rightarrow \)

\[
U(f,P) \leq U(f,P_1) \leq U(f) + \frac{\epsilon}{2} \quad \Rightarrow \quad U(f,P) - L(f,P) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

\( \Rightarrow L(f,P) \geq L(f,P_2) \geq L(f) - \frac{\epsilon}{2} \)

\( \therefore \) \( \Box \).