Examples (a) $f(x) = c = \text{const}$ on $[a,b]$.

In Example (a) p. 86 we showed that $L(f,P) = U(f,P) = c(b-a)$ for every partition P.

$\Rightarrow L(f) = U(f) = c(b-a)$

$\Rightarrow f$ is integrable, and $\boxed{\int_a^b c\,dx = c(b-a)}$

(b) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$ on $[a,b]$.

In Example (b) p. 86 we showed that $L(f,P) = 0$, $U(f,P) = b-a$ for every partition P.

$\Rightarrow L(f) = 0, \quad U(f) = b-a$

$\Rightarrow$ f is not integrable on $[a,b]$.

11/30/2011

A more user-friendly definition of integrability:

Proposition 32.5 (Criterion of integrability) A bounded function f on $[a,b]$ is integrable if and only if for each ε there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Moreover, ~~for such parti~~ if this holds then

$$L(f,P) < \int_a^b f(x)\,dx < L(f,P) + \varepsilon, \qquad U(f,P) - \varepsilon \le \int_a^b f(x)\,dx \le U(f,P).$$

Remark: Can state without ε: f is integrable $\Leftrightarrow$ $\exists$ sequence of partitions P_n s.t. $U(f,P_n) - L(f,P_n) \to 0$. In this case, $\int_a^b f(x)dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$

Proof ($\Leftarrow$: sufficiency) Let $\varepsilon > 0$, choose P so that $U(f,P) - L(f,P) < \varepsilon$.

Since by def. $U(f) \le U(f,P)$, $L(f) \ge L(f,P)$, we conclude that

$$U(f) - L(f) < \varepsilon.$$

Since this holds for ~~every~~ all $\varepsilon > 0 \Rightarrow U(f) - L(f) = 0 \Rightarrow f$ is integrable on $[a,b]$.

($\Rightarrow$: necessity) Suppose f is integrable $\Rightarrow U(f) = L(f) \Rightarrow$ by def,

$$\inf\{U(f,P)\} = \sup\{L(f,P)\}.$$

By def. of inf, sup: For every $\varepsilon > 0$ there exist partitions P_1, P_2 such that

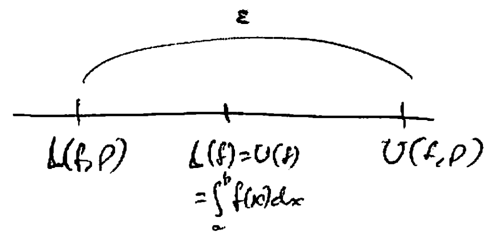
$$U(f,P_1) < U(f) + \varepsilon/2, \qquad L(f,P_2) > L(f) - \varepsilon/2.$$

Choose $P = P_1 \cup P_2$. By Prop. 32.2 (p. 86),

$$\left.\begin{aligned} U(f,P) &\le U(f,P_1) \le U(f) + \varepsilon/2 \\ L(f,P) &\ge L(f,P_2) \ge L(f) - \varepsilon/2 \end{aligned}\right\} \Rightarrow U(f,P) - L(f,P) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

* ~~Moreover~~ "Moreover" part: If f is integrable then

$$\int_a^b f(x)\,dx = U(f) = L(f).$$



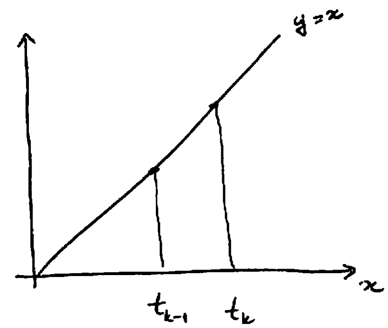
$L(f)$ lies between $L(f,P)$ and $U(f,P)$, and the dist. between $L(f,P)$, $U(f,P)$ is $<\varepsilon$.

$\Rightarrow$ the dist between $L(f)$ and each of $L(f,P)$, $U(f,P)$ is $\le \varepsilon$ QED.

~~Remark If (*) holds then~~

Examples

(a). $f(x) = x$ on $[0,1]$.



- Let P be any partition; since f is monotone we have

$$m(f,[t_{k-1},t_k]) = t_{k-1}, \quad M(f,[t_{k-1},t_k]) = t_k.$$

$$\Rightarrow L(f,P) = \sum_{k=1}^{n} t_{k-1}(t_k - t_{k-1}); \quad U(f,P) = \sum_{k=1}^{n} t_k (t_k - t_{k-1}).$$

$$\Rightarrow U(f,P) - L(f,P) = \sum_{k=1}^{n} (t_k - t_{k-1})^2 = ?$$

- Now choose P_n = even partition, where $t_k = \frac{k}{n}$, so $t_k - t_{k-1} = \frac{1}{n}$.

$$\Rightarrow U(f,P_n) - L(f,P_n) = \sum_{k=1}^{n} \left(\frac{1}{n}\right)^2 = \left(\frac{1}{n}\right)^2 \cdot n = \frac{1}{n} \to 0 \text{ as } n\to\infty.$$

By Prop. 32.5 (p. 88), $f(x)=x$ is integrable on $[0,1]$.

- To compute $\int_0^1 x\,dx$, use the "Moreover" part of Prop. 32.5. First compute

$$U(f,P) = \sum_{k=1}^{n} \frac{k}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n} \to \frac{1}{2} \text{ as } n\to\infty.$$

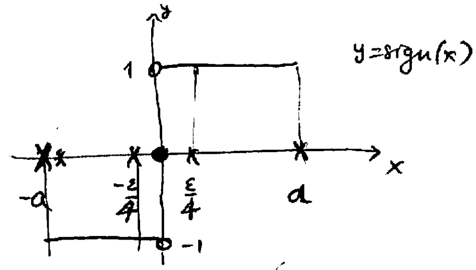
Since $\underbrace{U(f,P) - \frac{1}{n}}_{\downarrow\ 1/2} \le \int_0^1 x\,dx \le \underbrace{U(f,P)}_{\downarrow\ 1/2}$ $\Rightarrow$ by Squeeze Thm,

$$\boxed{\int_0^1 x\,dx = \frac{1}{2}.}$$

(b) $f(x) = \text{sign}(x)$ on $[-a, a]$.

Let $\varepsilon > 0$.

Choose the partition



$$P = \{-a, -\varepsilon/4, \varepsilon/4, a\}$$

$$\Rightarrow L(f,P) = (-1)\left(-\frac{\varepsilon}{4} + a\right) + (-1)\left(\frac{\varepsilon}{4} - \left(-\frac{\varepsilon}{4}\right)\right) + 1\left(a - \frac{\varepsilon}{4}\right) = -\varepsilon/2,$$

$$U(f,P) = (-1)\left(-\frac{\varepsilon}{4} + a\right) + 1\left(\frac{\varepsilon}{4} - \left(-\frac{\varepsilon}{4}\right)\right) + 1\left(a - \frac{\varepsilon}{4}\right) = \varepsilon/2$$

$$\Rightarrow U(f,P) - L(f,P) = \varepsilon$$

By Prop. 32.5, $\text{sign}(x)$ is integrable on $[a, a]$ and

$$\boxed{\int_{-a}^{a} \text{sign}(x)\,dx = 0}.$$

§

Riemann Sums (§32)

- The def. of integral is still ~~not~~ very convenient for applications.

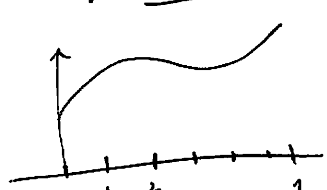
  Suppose f is integrable on $[a,b]$. To compute $\int_a^b f(x)\,dx$ using either Def. 32.1 (p. 87)

  or Criterion of Integrability (p. 88), we would need to find a good partition P

  for which $U(f,P) \approx L(f,P)$.

- Can we always use the even partition P as in Example (a) p. 87, and evaluate $f(x)$ at t_k instead of taking inf, sup;

Let's say $[0,1]$;



$$P = \left\{\frac{k}{n};\ k = 0, \dots, n\right\}$$

$$t_k - t_{k-1} = 1/n$$

$$\Rightarrow \boxed{\underbrace{\int_0^1 f(x)\,dx}_{\text{average}} = \lim_{n\to\infty} \frac{1}{n} \underbrace{\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)}_{\text{\& arithmetic mean}}\ ?} \quad \leftarrow \text{Is this true?} \qquad (*)$$

Yes! (if f is integrable; otherwise Dirichlet function provides a counterexample)