Examples \((a)\) \(f(x) = c = \text{const on } [a, b]\).

In Example \((a)\) p. 86 we showed that \(L(f, P) = U(f, P) = c(b-a)\) for any partition \(P\).

\[ L(f) = U(f) = c(b-a) \]

\[ \int_a^b c \, dx = c(b-a) \]

\(\Rightarrow\) \(f\) is integrable, and

\(8)\) \(f(x)=\begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}\) on \([a, b]\).

In Example \((a)\) p. 86 we showed that \(L(f, P) = 0, U(f, P) = b-a\) for any partition \(P\).

\[ L(f) = 0, \quad U(f) = b-a \]

\[ \Rightarrow f\] is not integrable on \([a, b]\).

A more user-friendly definition of integrability:

**Proposition 32.5 (Criterion of integrability).** A bounded function \(f\) on \((a, b)\) is integrable if and only if for each \(\varepsilon > 0\) there exists a partition \(P\) of \([a, b]\) such that

\[ U(f, P) - L(f, P) < \varepsilon. \]

Moreover, for such a partition, it holds that

\[ L(f, P) < \int_a^b f(x) \, dx < U(f, P) + \varepsilon, \quad \varepsilon \leq \int_a^b f(x) \, dx \leq U(f, P). \]

**Proof** (\(\Leftarrow\) : sufficiency) \(\Rightarrow\) \(\exists\) \(\varepsilon > 0\), choose \(P\) so that \(U(f, P) - L(f, P) < \varepsilon\).

Since by def. \(U(f) = U(f, P),\) \(L(f) = L(f, P)\), we conclude that

\[ U(f) - L(f) < \varepsilon \]

Since this holds for all \(\varepsilon > 0 \Rightarrow U(f) - L(f) = 0 \Rightarrow f\) is integrable on \([a, b]\).

(\(\Rightarrow\) : necessity) Suppose \(f\) is integrable \(\Rightarrow U(f) = L(f) \Rightarrow \) by def.

\[ \inf \{ U(f, P) \} = \sup \{ L(f, P) \}. \]

By def. of \(\inf, \sup\): For every \(\varepsilon > 0\) there exist partitions \(P_1, P_2\) such that

\[ U(f, P_1) < U(f, P_2) + \varepsilon/2, \quad L(f, P_2) < L(f, P_1) - \varepsilon/2. \]

Choose \(P = P_1 \cup P_2\). By Prop 32.2 (p. 86),

\[ U(f, P) \leq U(f, P_1) \leq U(f) + \varepsilon/2 \]
\[ L(f, P) \geq L(f, P_2) \geq L(f) - \varepsilon/2 \]

\[ \Rightarrow U(f, P) - L(f, P) \leq \varepsilon \frac{1}{2} \leq \frac{\varepsilon}{2} = \varepsilon. \]
*Moreover* part: If \( f \) is integrable then
\[
\int_a^b f(x) \, dx = U(f) = L(f).
\]

\( L(f) \) lies between \( L(f, P) \) and \( U(f, P) \), and the dist. between \( L(f, P), U(f, P) \) is \( \epsilon \).

\( \Rightarrow \) the dist between \( L(f) \) and each of \( L(f, P), U(f, P) \) is \( \leq \epsilon \) \( \forall \epsilon > 0 \).

**Examples**

(a) \( f(x) = x \) on \([0, 1]\).

Let \( P \) be any partition; since \( f \) is monotone we have
\[
m(t, [t_k, t_{k+1}]) = t_k, \quad M(t, [t_k, t_{k+1}]) = t_{k+1}.
\]

\( \Rightarrow \)
\[
L(f, P) = \sum_{k=1}^{n} t_k \quad (t_k - t_{k-1}); \quad U(f, P) = \sum_{k=1}^{n} t_k \quad (t_k - t_{k-1}).
\]

\( \Rightarrow \)
\[
U(f, P) - L(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1})^2 = \frac{1}.\]

Now choose \( P = \) even partition, where \( t_k = \frac{k}{n} \), so \( t_k - t_{k-1} = \frac{1}{n} \).

\( \Rightarrow \)
\[
U(f, P) - L(f, P) = \sum_{k=1}^{n} \left( \frac{1}{n} \right)^2 = \frac{1}{n}.\quad n = \frac{1}{n} \quad \to 0 \text{ as } n \to \infty.
\]

By Prop. 32.5 (88), \( f(x) = x \) is integrable on \([0, 1]\).

To compute \( \int_0^1 x \, dx \), use the *Moreover* part of Prop. 32.5. First compute
\[
U(f, P) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n} \to \frac{1}{2} \text{ as } n \to \infty.
\]

Since \( U(f, P) - \frac{1}{n} \leq \int_0^1 x \, dx \leq U(f, P) \), \( \Rightarrow \) by Squeeze Theorem,
\[
\int_0^1 x \, dx = \frac{1}{2}.
\]
(b) \( f(x) = \text{sign}(x) \) on \([-a, a]\).

Let \( \varepsilon > 0 \).

Choose the partition
\[
P = \{-a, -\frac{\varepsilon}{4}, \frac{\varepsilon}{4}, a\}.
\]

\[
\Rightarrow \quad L(f, P) = (-1)(-\frac{\varepsilon}{4} + a) + (-1)(\frac{\varepsilon}{4} - (-\frac{\varepsilon}{4})) + 1(a - \frac{\varepsilon}{4}) = -\varepsilon/2,
\]

\[
U(f, P) = (-1)(-\frac{\varepsilon}{4} + a) + 1(\frac{\varepsilon}{4} - (-\frac{\varepsilon}{4})) + 1(a - \frac{\varepsilon}{4}) = \varepsilon/2.
\]

\[
\Rightarrow \quad U(f, P) - L(f, P) = \varepsilon.
\]

By Prop. 32.5, \( \text{sign}(x) \) is integrable on \([-a, a]\) and
\[
\int_{-a}^{a} \text{sign}(x) \, dx = 0.
\]

\[3\]

Riemann Sums (§32)

- The def. of integral is still not very convenient for applications.
- Suppose \( f \) is integrable on \([a, b]\). To compute \( \int_{a}^{b} f(x) \, dx \) using either Def. 32.1 (p. 87) or Criterion of Integrability (p. 88), we would need to find a good partition \( P \) for which \( U(f, P) = L(f, P) \).
- Can we always use the even partition \( P \) as in Example (x) P. 87, and evaluate \( f(x) \) at \( t \) instead of taking \( \frac{t}{n} \), sup:\n
Let's say \([0, 1]\):
\[
P = \left\{ \frac{k}{n}, \frac{k+1}{n}, \ldots \right\}
\]

\[
t_k - t_{k-1} = \frac{1}{n}
\]

\[
\Rightarrow \quad \int_{0}^{1} f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(t_k) ?
\]

- Is this true? (x)

Yes! \((\text{if } f \text{ is integrable; otherwise Dirichlet function provides a counterexample})\)