Theorem 32.7 (Mesh) For a partition $P = \{t_0, t_1, \ldots, t_k\}$, let $\text{mesh}(P) = \max \{t_k - t_{k-1}\}$. A bounded function $f$ on $[a, b]$ is integrable if and only if:

for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$\text{mesh}(P) < \delta$ implies $U(f, P) - L(f, P) < \varepsilon$.

Remarks
(a) This can be stated in a sequential form:

$$f \text{ is integrable if and only if:}$$

for every sequence of partitions $P_n$ with $\text{mesh}(P_n) \to 0$,

one has $U(f, P_n) - L(f, P_n) \to 0$.

In this case, by Prop. 32.5 (p. 88) (Criterion of Integrability),

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).$$

Exercise: Prove (a).

(b) Thm 32.7 allows one to use even partitions in computing integrals.

---

Proof of Thm. ($\Leftarrow$: Sufficiency). Choose any $P_n$ with $\text{mesh}(P_n) \to 0$.

Then $U(f, P_n) - L(f, P_n) \to 0$, so $f$ is integrable by the criterion (Prop. 32.5 p. 88).

($\Rightarrow$: Necessity). Suppose $f$ is bounded, integrable; let $\varepsilon > 0$, choose $\delta = \delta(\varepsilon)$.

Let $P$ be a partition with mesh $(P) < \delta$.

WTS: $U(f, P) - L(f, P) < \varepsilon$. 

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Since $f$ is integrable, there exists a partition $P_0$ such that
\[ U(f, P_0) - L(f, P_0) < \frac{\epsilon}{2}. \]

Want to transfer this property to $P_0'$. Combine: $Q := P_0 P_0'$. By Prop. 32.2, $L(f, P_0) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_0)$, so all four numbers are at most $\epsilon/2$ apart.

It remains to show that
\[ L(f, Q) - L(f, P) < \frac{\epsilon}{4}, \quad U(f, P) - U(f, P_0) < \frac{\epsilon}{4}. \]

**Claim:**
\[ L(f, Q) - L(f, P) \leq m \cdot 2B \cdot \text{mesh}(P) \]

where $m = \#$ points in $P_0$; $|f(x)| \leq B$ on $[a, b]$.

Indeed, each point in $Q$ that is not in $P$ changes one term in the sum for $L(f, P)$, which is of the form $m \cdot \frac{(t_k - t_{k-1})}{\text{mesh}(P)}$ between $-B, B \leq \text{mesh}(P)$.

So by the triangle inequality,
\[ U(f, P) - L(f, P) < \epsilon \]

So we choose $\delta$ so that $m \cdot 2B \cdot \text{mesh}(P) < \frac{\epsilon}{4}$, i.e., set $\delta = \frac{\epsilon}{8mB}$.

So it suffices to have $m \cdot 2B \cdot \text{mesh}(P) < \frac{\epsilon}{4}$, i.e., $\text{mesh}(P) < \frac{\epsilon}{8mB}$.

Set $\delta = \frac{\epsilon}{8mB}$; since $\text{mesh}(P) < \delta$, Q.E.D.
**Definition 3.2** (Riemann sums).

Let \( f \) be a bounded function on \([a, b]\), and \( P \) be a partition of \([a, b]\).

A **Riemann sum** is a sum of the form

\[
\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) \quad \text{where} \quad x_k \in [t_{k-1}, t_k].
\]

---

**Theorem 3.2.9** (Riemann sums). Let \( f \) be an integrable function on \([a, b]\).

Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( P \) is a partition of \([a, b]\) with \( \text{mesh}(P) < \delta \),

\[
\left| \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) - \int_{a}^{b} f(x) \, dx \right| < \varepsilon
\]

whenever \( x_k \in [t_{k-1}, t_k] \) are arbitrary

\( S := \text{Riemann sum} \).

---

**Proof**

Choose \( \delta = \delta(\varepsilon) \) as in Theorem 3.2.7 (p. 91); then

\[
U(f, P) - L(f, P) < \varepsilon.
\]

Clearly, both \( \int_{a}^{b} f(x) \, dx \) and the Riemann sum \( S \) lie between \( L(f, P) \) and \( U(f, P) \).

\[
\Rightarrow \quad \left| \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) - \int_{a}^{b} f(x) \, dx \right| < \varepsilon.
\]

Q.E.D.

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**Remarks**

(a) (Sequential form of Thm 3.2.9):

Let \( f \) be integrable on \([a, b]\). Then, for any sequence of partitions \( P_n \) of \([a, b]\) with \( \text{mesh}(P_n) \to 0 \) and any sequence of Riemann sums \( S_n \) associated with them,

\[
S_n \to \int_{a}^{b} f(x) \, dx.
\]
(b) Converse also holds in Thm 3.2.9. If the conclusion of remark (a) holds then $f$ is integrable on $[a, b]$. (See Thm 3.2.9 in Ross.)

(c) Computational aspect:

Thm 3.2.9 allows us to compute integrals using any partitions (fine enough) and any points $x_k$ of our choice.

In particular:

Cor: Let $f$ be integrable on $[a, b]$. Then

$$\int_0^1 f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right)$$

Proof: Use the set over partitions $P_n = \{ \frac{k}{n}, \frac{k+1}{n}, \ldots, \frac{n}{n} \}$ and $x_k = \frac{k}{n}$.

$\quad$ Q.E.D.

Example: $\sum_{k=1}^{n} k^2 = ?$ (Asymptotically).

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{k}{n} \right)^2 = \int_0^1 x^2 \, dx = \frac{x^3}{3} \bigg|_0^1 = \frac{1}{3} \implies \sum_{k=1}^{n} k^2 \approx \frac{n^3}{3} \quad \text{as } n \to \infty.$$  

Note that the exact sum is $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (Ex.1.1 Ross).