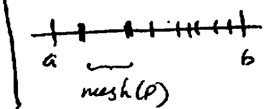
12/02/2011

Theorem 32.7 (Mesh) For a partition $P=\{t_k\}$, let $\text{mesh}(P)=\max_k\{t_k-t_{k-1}\}$.

A bounded function f on $[a,b]$ is integrable if and only if:
for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\text{mesh}(P)<\delta \quad \text{implies} \quad U(f,P)-L(f,P)<\varepsilon.$$



Remarks (a) Thm can be stated in a sequential form:

(*) f is integrable if and only if:
for every sequence of partitions P_n with $\boxed{\text{mesh}(P_n)\to 0,}$
one has $U(f,P_n)-L(f,P_n)\to 0$.

In this case by Prop. 32.5 (p.88) (Criterion of integrability),

$$\int_a^b f(x)\,dx = \lim_n U(f,P_n) = \lim_n L(f,P_n).$$

Exercise: prove (*).

(b). Thm 32.7 allows one to use even partitions  in computing integrals.

Proof of Thm. ($\Leftarrow$: sufficiency). Choose any P_n with $\text{mesh}(P_n)\to 0$.
Then $U(f,P_n)-L(f,P_n)\to 0$, so f is integrable by the criterion (Prop. 32.5 p.88)

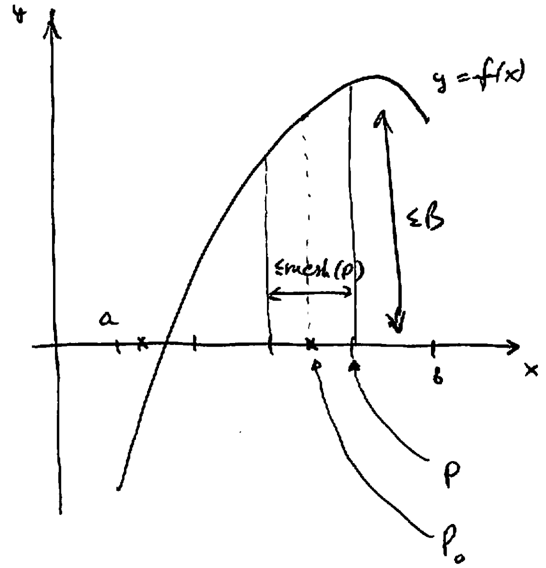
($\Rightarrow$: Necessity). Suppose f is bounded, integrable; let $\varepsilon>0$, choose $\delta=\delta(\varepsilon)$ TBD.
Let P be a partition with $\text{mesh}(P)<\delta$.

WTS: $U(f,P)-L(f,P)<\varepsilon$.

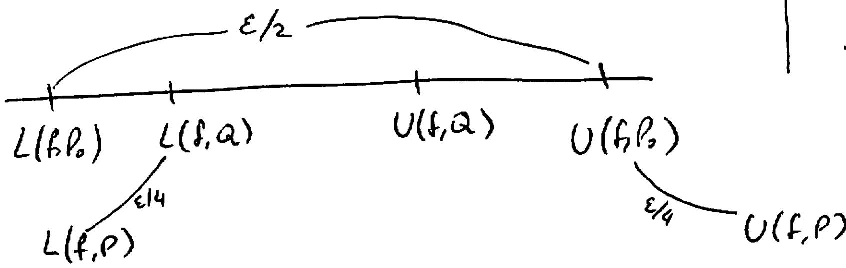
Since f is integrable, there exists a partition P_0 such that

$$U(f,P_0) - \underset{\uparrow}{L}(f,P_0) < \frac{\varepsilon}{2}.$$

Want to transfer this property to P.



- Combine: $\quad Q := P \cup P_0$.



By Prop. 32.2, $\quad L(f,P_0) \le L(f,Q) \le U(f,Q) \le U(f,P_0)$,

so all four numbers are at most $\varepsilon/2$ apart.

It remains to show that

$$L(f,Q) - L(f,P) < \frac{\varepsilon}{4}, \qquad U(f,P) - U(f,P_0) < \varepsilon/4.$$

(see Figure, apply triangle inequality — Ex.)

<u>Claim</u>: $L(f,Q) - L(f,P) \le m \cdot 2B \cdot \text{mesh}(P)$

where $m = \#$ points in P_0; $\quad |f(x)| \le B$ on $[a,b]$.

(Indeed, each of the m points in Q that is not in P changes one term in the sum for $L(f,P)$, which is of the form $\underset{\text{between } -B,\,B}{m_k}\,\underbrace{(t_k - t_{k-1})}_{\le \text{mesh}(P)}$)

~~So by the triangle inequality,~~

~~$U(f,P) - L(f,P) \le \varepsilon$~~

~~So we choose δ so that $m \cdot 2B \cdot \text{mesh}(P) < \frac{\varepsilon}{4}$, ie set $\delta =$~~

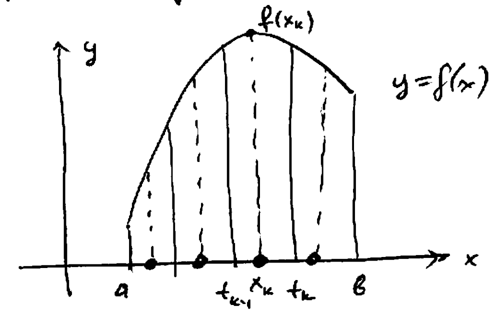
So it suffices to have $\quad m \cdot 2B \cdot \text{mesh}(P) < \varepsilon/4 \iff \text{mesh}(P) < \dfrac{\varepsilon}{8mB}$.

Set $\delta = \dfrac{\varepsilon}{8mB}$; since $\text{mesh}(P) < \delta$, QED.

Def 32.8 (Riemann sums).

Let f be a bounded function on $[a,b]$, and P be a partition of $[a,b]$.
A <u>Riemann sum</u> is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) \quad \text{where } x_k \in [t_{k-1}, t_k].$$

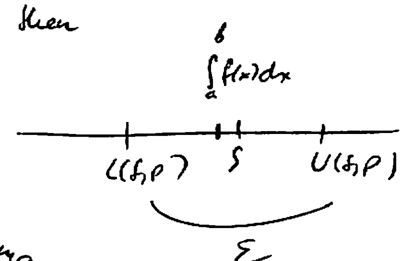


Theorem 32.9 (Riemann sums). Let f be an integrable function on $[a,b]$.
Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that ~~every pa~~
whenever P is a partition of $[a,b]$ with $\text{mesh}(P) < \delta$,

$$\Big| \underbrace{\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})}_{S := \text{Riemann sum}} - \int_a^b f(x)\,dx \Big| < \varepsilon \qquad , \text{ where } x_k \in [t_k, t_{k-1}] \text{ are arbitrary.}$$

<u>Proof</u> ~~Because~~, Choose $\delta = \delta(\varepsilon)$ as in Theorem 32.7 (p. 91); then

$$U(f,P) - L(f,P) < \varepsilon.$$



~~Clearly, bo~~ Clearly, both $\int_a^b f(x)\,dx$ and the Riemann sum S lie between $L(f,P)$ and $U(f,P)$.

$$\Rightarrow \quad \Big| S - \int_a^b f(x)\,dx \Big| < \varepsilon.$$

QED.

<u>Remarks</u> (a). (Sequential ~~~~ form of Thm 32.9):

Let f be integrable on $[a,b]$. Then, for any sequence of partitions P_n of $[a,b]$ with $\text{mesh}(P_n) \to 0$ and any ~~sequence of~~ Riemann sums S_n associated with them,

$$S_n \to \int_a^b f(x)\,dx.$$

(b) Converse also holds in Thm 32.9: if the conclusion of remark (a) holds then f is integrable on $[a,b]$. (See Thm 32.9 in Ross).

(c) Computational aspect:

Thm 32.9 allows us to compute integrals using any partitions (fine enough) and any points x_k of our choice.

In particular:

$$\boxed{\textbf{Cor}\ \ \text{Let } f \text{ be integrable on } [a,b]. \text{ Then } \int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)}$$

~~Proof: Use the even partition $P_n = \{\frac{k}{n} : k=1,\dots n\}$ and $x_k =$~~
~~In RHS, these are Riemann su~~

Proof: The RHS is the limit of Riemann sums of f corresponding to the even partitions ~~$P_n = \{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$~~ $P_n = \{\frac{k}{n} : k=1 \dots n\}$ and $x_k = \frac{k}{n}$. QED.

Example: $\sum_{k=1}^{n} k^2 = ?$ (Asymptotically).

$$\underbrace{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2}_{= \frac{1}{n^3}\sum_{k=1}^{n} k^2} = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}. \quad \Rightarrow \quad \boxed{\sum_{k=1}^{n} k^2 \approx \frac{n^3}{3}} \text{ as } n\to\infty.$$

Note that the exact sum is $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (Ex. 1.1 Ross)