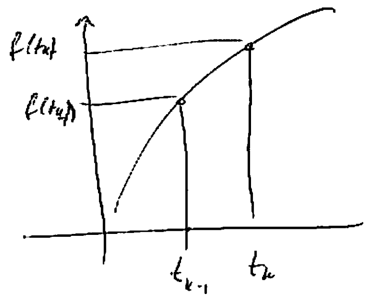
§33. Properties of Riemann Integral

Thm 33.1 Every monotonic function f on $[a,b]$ is integrable.

Proof ~~Assume~~ WLOG. f is non-decreasing.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(t_k - t_{k-1}),$$

where $M_k = M(f,[t_{k-1}, t_k])$, $m_k = m(f,[t_{k-1}, t_k])$.



$$= \sum_{k=1}^{n} \underbrace{\left(f(t_k) - f(t_{k-1})\right)}_{\text{(by monotonicity)}} \underline{(t_k - t_{k-1})}$$

$$\le \text{mesh}(P) \cdot \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right) \quad \leftarrow \text{telescoping}$$

$$= \text{mesh}(P) \cdot \left(f(b) - f(a)\right).$$

$\Rightarrow$ f is integrable by Thm 32.7 (p.91) (with $\delta = \frac{\varepsilon}{f(b) - f(a)}$). QED.

Thm 33.2. Every continuous function f on $[a,b]$ is integrable.

Proof f is continuous $\Rightarrow$ f is uniformly continuous on $[a,b]$ (Thm 19.2)

Hence for each $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|x-y| < \delta, \quad x, y \in [a,b] \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$, choose δ as above. Consider any partition P with mesh$(P) < \delta$.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \underbrace{(M_k - m_k)}_{\wedge\ \varepsilon}(t_k - t_{k-1})$$

ε (because f attains max, min on $[t_{k-1}, t_k]$ by Theorem 18.1)

$$\le \varepsilon \sum_{k=1}^{n} (t_k - t_{k-1}) = \varepsilon(b-a).$$

Hence f is integrable by Thm 32.7 (p.91). QED.

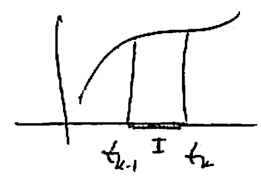
Thm 33.3 (Sums, products)

Let f, g be integrable on $[a,b]$. Then:

(i) $\forall k \in \mathbb{R}$, kf is integrable and $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$;

(ii) $f+g$ is integrable and $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$;

(iii) fg is integrable.



Proof (i) – Ex. (iii) – HW Ex. 33.8.

(ii): Let P be a partition on $[a,b]$.

We want to compare $U(f+g)$ to $U(f)+U(g)$, and $L(f+g)$ to $L(f)+L(g)$.

~~To this en~~ Recalling the def. of these upper/lower sums, we ~~need to~~ see that we need to compare $M(f+g, I)$ to $M(f,I)+M(g,I)$ where $I = [t_{k-1}, t_k)$, and similarly ~~for~~ $m(f+g, I)$ to $m(f,I)+m(g,I)$

- $$M(f+g, I) = \sup\{f(x)+g(x) : x \in I\} \overset{\text{by def. of sup}}{\leq} \sup\{f(x) : x\in I\} + \sup\{g(x) : x \in I\}$$
$$= M(f,I) + M(g,I).$$

Similarly, $m(f+g, I) \geq m(f,I) + g(I)$.

- $\Rightarrow$
$$U(f+g,P) \leq U(f,P) + U(g,P) \qquad (*)$$
$$\overset{\vee\!|}{L}(f+g,P) \geq L(f,P) + L(g,P). \qquad (**)$$

- Now let P_n be any sequence of partitions with $\text{mesh}(P_n) \to 0$.

~~RHS of (*) $\Rightarrow \int_a^b f(x)dx$~~

Right hand sides of $(*)$, $(**)$ converge to $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (by Thm 32.7 (Mesh) p.91)

$\Rightarrow$ by Squeeze Theorem,

$U(f+g, P_n)$ and $L(f+g, P_n)$ converge to the same quantity.

$\Rightarrow$ Again by Thm 32.7, QED.

Estimates of an integral :

Proposition If f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, then
$$\int_a^b f(x)\,dx \geq 0$$

Proof Since $f \geq 0$, $U(f,P) \geq 0$ and $L(f,P) \geq 0$ for any partition P.
The conclusion of Prop. follows by def. of $\int$. QED.

(33.4) (Comparison of integrals)
Corollary If f, g are integrable on $[a,b]$ and
$$f(x) \leq g(x) \quad \text{for all } x \in [a,b]$$
then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof The function $h := g - f$ satisfies the assumptions of Proposition (by Thm. on the sum of integrals p. 96)
$$\Rightarrow \int_a^b (g-f)(x) \geq 0 \Rightarrow \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0 \quad \text{by the same Prop} \qquad \text{QED.}$$

Cor If f is integrable and $m \leq f(x) \leq M$ for all $x \in [a,b]$ then $\boxed{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}$ (Proof: use for $g(x) = \text{const}$).

Theorem 33.5 (Absolute Value) If f is integrable on $[a,b]$ then $|f|$ is integrable on $[a,b]$ and
$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx. \qquad (*)$$

Remark: This result is delicate; The converse is not true: $\exists\, |f|$ integrable, f non-integrable (HW Ex. 33.4).

Proof 1) If we know that $|f|$ is integrable, $(*)$ follows easily from Cor. above, as $-|f(x)| \leq f(x) \leq |f(x)|$
$$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \qquad \Rightarrow \text{QED}$$

2) Proof that $|f|$ is integrable.