Estimates of an integral

Prop If f is integrable on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$ then
$$\int_a^b f(x)\,dx \geq 0 .$$

Proof $f \geq 0$ implies $U(f,P) \geq 0$, $L(f,P) \geq 0$ for any partition P.

~~By def of~~ $\overset{\text{def.}}{\Longrightarrow}$ $\int_a^b f(x)\,dx = \inf_P U(f,P) \geq 0$ QED.

Cor 33.4 (Comparison) If f, g are integrable on $[a,b]$ and
$$f(x) \leq g(x) \quad \text{for all } x \in [a,b)$$
then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx .$$

Proof Apply Prop. for $h = g - f$. QED.

$\boxed{12/07/2011}$

Cor ~~[illegible]~~ If f is integrable on $[a,b]$ and
$$m \leq f(x) \leq M \qquad \text{for } x \in [a,b]$$
then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Rewrite as
$$m \leq \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{} \leq M$$
↑ average of f on $[a,b]$.

Use for $m = \min_{x\in[a,b]} f(x)$, $M = \max_{x\in[a,b]} f(x)$ along with I.V.T. $\Longrightarrow$

Thm 33.9 (Intermediate Value Thm for functions) Let f be continuous on $[a,b]$.
Then $\exists\, x_0 \in [a,b]$ such that
$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)\,dx .$$

Thm 33.5 If f is integrable on $[a,b]$ ~~then~~ $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx. \qquad (*)$$

Remark: The converse is not true: $\exists |f|$ integrable, f non-integrable (Ex. 33.4 HW)

Proof 1) If we know that $|f|$ is integrable then $(*)$ follows from Cor. 33.4 (Comparison):

Since $-|f(x)| \le f(x) \le |f(x)|$ for all x,

$$\Rightarrow \quad -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx, \qquad \text{QED.}$$

2) Proof that $|f|$ is integrable: follows from

$$U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$$

which itself clearly follows from

$$M(|f|,I) - m(|f|,I) \le M(f,I) - m(f,I) \qquad (\forall \text{ set } I)$$

which is Ex. 33.6 HW.

QED.

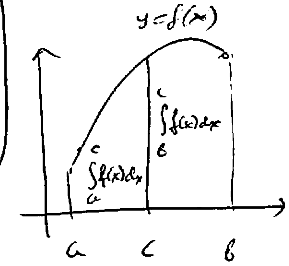
Integral is ~~an~~ additive with respect to the interval of integration: [illegible]/2011

Thm 33.6 Let f be a function on $[a,b]$.

If $a<c<b$ and f is integrable on $[a,c]$ and on $[c,b]$,

then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx .$$

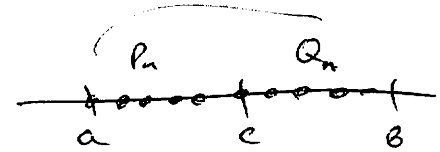


Proof Follows from the corresponding ~~addit~~ property of ~~upper/lower~~ integral sums.

Indeed, ~~since f is into~~ since the restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$ are integrable, there exist sequences of partitions (P_n) of $[a,c]$ and (Q_n) of $[c,b]$ such that:

$$\lim_n L(f|_{[a,c]}, P_n) = \lim_n U(f|_{[a,c]}, P_n) \Rightarrow \int_a^c f(x)\,dx \;;$$

$$\lim_n L(f|_{[c,b]}, Q_n) = \lim_n L(f|_{[c,b]}, Q_n) = \int_c^b f(x)\,dx$$



~~Sum~~ Add these, and note that

~~Each~~

$$L(f|_{[a,c]}, P_n) + L(f|_{[c,b]}, Q_n) = L(f, P_n \cup Q_n),$$

$$U(\ \text{——}\ ) + U(\ \text{——}\ ) = U(\ \text{——}\ ).$$

$$\Rightarrow \lim_n L(f, P_n \cup Q_n) = \lim_n U(f, P_n \cup Q_n) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

By criterion of integrability, Q.E.D.

Rem Thm 33.6 motivates the following conventions:

- $\int_a^a f(x)\,dx = 0$;
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, $b>a$.

(33.8) Thm If f is ~~bounded and~~ piecewise continuous (i.e. finitely many pts of discontinuity) on $[a,b]$ ~~or piecewise monotone on [a,b]~~

then f is integrable on $[a,b]$.

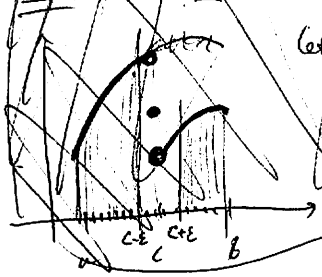
Examples: (a) $f(x) = \operatorname{sign} x$, (b) $g(x) = [x]$, are bounded piecewise-continuous, ~~piecewise monotone~~ on $\mathbb{R}$ / $[a,b]$.

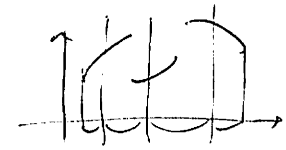
~~(c) Ex: give an example of a piecewise-monotone but not piecewise-monotone function on [a,b]~~

Proof (for one pt of discontinuity — give a proof for general case ~~d pts of~~)

Let $\varepsilon>0$, let ~~P_n~~ mesh$(P_n) \to 0$.

Then



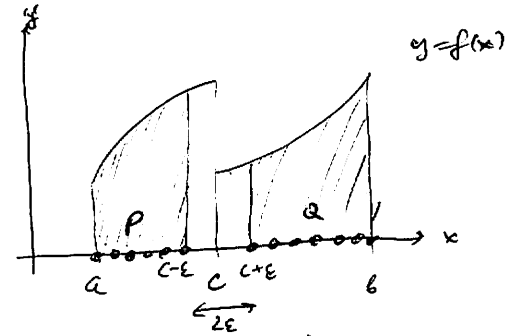


Proof: Will give a proof for f that has one pt of discontinuity. The general case will then follow from Thm. 33.6 (how?)

Let $\varepsilon > 0$; then $f|_{[a,c-\varepsilon]}$ and $f|_{[c+\varepsilon,b]}$ are both continuous $\Rightarrow$ integrable $\Rightarrow$ $\exists\, P$ of $[a,c-\varepsilon]$ and Q of $[c+\varepsilon,b]$ such that

$$U(f|_{[a,c-\varepsilon]}, P) - L(f|_{[a,c-\varepsilon]}, P) < \varepsilon,$$

$$U(f|_{[c+\varepsilon,b]}, Q) - L(f|_{[c+\varepsilon,b]}, Q) < \varepsilon.$$



Add them; noting that

$$U(f, P\cup Q) = U(f|_{[a,c-\varepsilon]}, P) + U(f|_{[c+\varepsilon,b]}, Q) + \underbrace{M(f,[c-\varepsilon,c+\varepsilon])\cdot 2\varepsilon}_{\text{this is term of the upper sum}}$$

and similarly

$$L(f, P\cup Q) = L(\ - - - \ ) + L(\ - - - \ ) + m(\ - - - \ )\cdot 2\varepsilon.$$

$$\Rightarrow U(f,P\cup Q) - L(f,P\cup Q) < \varepsilon + \varepsilon + 2B\cdot 2\varepsilon \qquad \text{~~because~~ where } |f(x)|\le B \text{ for all } x\in[a,b]$$

$$\le (2+4B)\varepsilon.$$

ε arbitrary $\Rightarrow$ f is integrable by the Criterion of Integrability. QED

Thm (Lebesgue Criterion) f is integrable on $[a,b]$ is (Riemann) integrable on $[a,b]$ if and only if f is bounded on $[a,b]$ and is continuous on $[a,b]$ almost everywhere

means: "the set I of points of discontinuity of f has measure 0"

means $\forall\, \varepsilon > 0$ I can be covered by a union of countably many intervals I_k (i.e. $I \subset \bigcup_{k=1}^{\infty} I_k$) whose sum of lengths $\sum_{1}^{\infty} |I_k| < \varepsilon$.

- In particular, ~~f is allow~~ any countable set of discontinuities has measure 0 $\Rightarrow$ OK for integrability. (Ex: prove directly!)