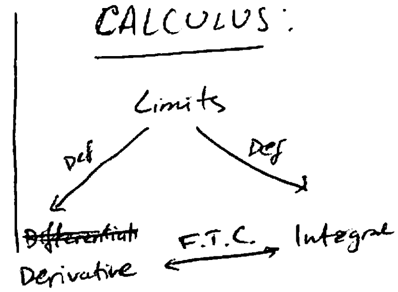
12/09/2011

§34. Fundamental Thm of Calculus.



= "Integration and differentiation are opposite operations".

~~= ∃ Int~~ $=\begin{cases} \text{I. Integral of the derivative of a function} = \text{function} \\ \text{II. Derivative of the integral of a function} = \text{function.} \end{cases}$

F.T.C. I (34.1). ~~Let~~ f be differentiable on $[a,b]$.
If f is integrable on $[a,b]$ then
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Proof • g' integrable $\overset{\forall \varepsilon>0}{\Rightarrow}$ ∃ partition ~~P of~~ $P = \{t_k\}$ of $[a,b]$ s.t.
$$\left| \sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) - \int_a^b f'(x)\,dx \right| < \varepsilon \quad \text{for } \forall \text{ choice of pts } x_k \in [t_{k-1}, t_k]$$

(see Riemann Sums)

• Use MVT on: ∃ $x_k \in [t_{k-1}, t_k]$ such that
$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} \quad \Rightarrow \quad f'(x_k)(t_k - t_{k-1}) = f(t_k) - f(t_{k-1})$$

$\Rightarrow$ for these x_k,
$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = f(b) - f(a)$$

$$\Rightarrow \quad \left| f(b) - f(a) - \int_a^b f'(x)\,dx \right| < \varepsilon$$

$\varepsilon > 0$ arbitrary $\Rightarrow$ QED

Remark ∃ functions f that are differentiable on $[a,b]$ but s.t. f' is is not bounded ($\Rightarrow$ not integrable) on $[a,b]$, e.g.
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{on } [0,1]$$

(Ex – check!)

~~Remark We can write FTC I as:~~ $\cancel{\int_a^x f'(x)\,dx = f(x) - f(a) \quad \forall x}$

$\cancel{\int_a^x g(x)\,dx = G(x) - G(a)}$ ~~where~~ $\cancel{G'(x) = g(x)}$

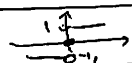
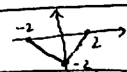
$\cancel{G(x) = \int_a^x g(x) + G(a).}$

~~⇒ Antiderivative G of g exists and is given by~~ $\boxed{G(x) = \int_a^x g(x) + C.}$ $\int g(x)\,dx$ Indef. integral

F.T.C. II (34.3) Let f be integrable on $[a,b]$, and consider the function
$$F(x)=\int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$.

Moreover, if f is continuous at $x_0\in(a,b)$ then F is differentiable at x_0, and
$$F'(x_0)=f(x_0).$$

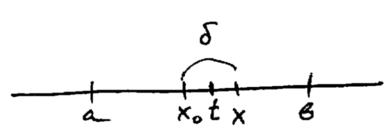
Example: $f(t)=\mathrm{sign}(t)$;  $a=-2$; $F(x)=$ 

Note that F is differentiable exactly where f is continuous, i.e. everywhere except 0.

~~Proof~~ ~~To show that F is conti~~

Proof 1) To show that F is continuous, consider $x,y\in[a,b]$; $x\le y$ $\Rightarrow$
$$|F(x)-F(y)|=\left|\int_x^y f(t)\,dt\right|\le\int_x^y|f(t)|\,dt \qquad \text{(by Thm 33.5)}$$
$$\le\int_x^y B\,dt \qquad \left(\text{if } |f(x)|\le B \text{ for all } x\in[a,b];\ \text{since } f \text{ is integrable} \Rightarrow \text{bounded} \Rightarrow \exists \text{ such } B\right)$$
$$=B(y-x)=B|x-y|.$$
This shows that F is (uniformly) continuous (with $\delta=\varepsilon/B$ in the def. of continuity).

2) "Moreover" part. $F'(x_0)$ is the limit, as $x\to x_0$, of
$$\frac{F(x)-F(x_0)}{x-x_0}=\underbrace{\frac{1}{x-x_0}\int_{x_0}^{x}f(t)\,dt}_{\text{average}}$$
we want to bound $\Rightarrow$
$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|=\left|\frac{1}{x-x_0}\int_{x_0}^{x}[f(t)-f(x_0)]\,dt\right|$$
$$\le\frac{1}{|x-x_0|}\int_{x_0}^{x}|f(t)-f(x_0)|\,dt \quad (*) \quad \left(\text{here let's assume } x_0<x;\ \text{the opposite case is similar - Ex.}\right)$$
Now we use the continuity of f to bound this. Let $\varepsilon>0$;
$\exists\,\delta>0$: $|t-x_0|<\delta$ implies $|f(t)-f(x_0)|<\varepsilon$.



If $|x-x_0|<\delta$ then $|t-x_0|<\delta \Rightarrow |f(t)-f(x_0)|<\varepsilon \Rightarrow$
$(*)\le\varepsilon$. (because in $(*)$ we have the average of $|f(t)-f(x_0)|$; see Cor. p. 97)

$$\Rightarrow \frac{F(x)-F(x_0)}{x-x_0}\to f(x_0) \text{ as } x\to x_0.$$

Antiderivative and indefinite integral

Def Let f be a function on $[a,b]$.
An antiderivative of f is a function F on (a,b) such that
$$F'(x) = f(x) \quad \text{for all } x \in (a,b).$$

THM (Antiderivative). Let f be a function on $[a,b]$.

1. Existence : If f is continuous ~~on $[a,b]$~~ then an antiderivative of f exists, and is given by
$$F(x) = \int_a^x f(t)\,dt .$$

2. Uniqueness : The antiderivative is unique up to an additive const C.
Namely, if F, G are two antiderivatives of f then ~~$F'(x) = G'(x) = f(x)$ for all $x \in (a,b)$~~
then ~~$F(x) = G(x) + C$~~ $F = G + C$.

Proof (1) follows from F.T.C II.
(2) follows from Cor. p. 72 (to Mean Value Thm): if $F' = G' = f$ then
$$(F-G)' = 0 \Rightarrow F - G = \text{const}.$$
QED.

- Thm. implies that the general form of antiderivative of f is given by
$$\boxed{F(x) = \int_a^x f(t)\,dt + C} \quad \text{where } a, C \text{ are constants.} \qquad (*)$$

Moreover, by F.T.C I
$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} \qquad (**)$$

Def The indefinite integral of f is the collection of all antiderivatives of f (which all differ by const).

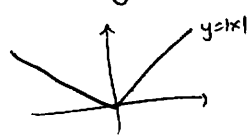
~~The indefinite~~ In view of (*), the indefinite integral of f is written as

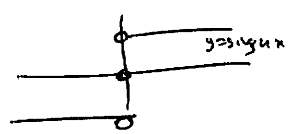
$$\boxed{\int f(x)\,dx + C}\ .$$

Examples. (a) $\int \cos x\,dx = \sin x + C$ means that:

- $(\sin x)' = \cos x$, (ie. $\sin x$ is the antiderivative of $\cos x$;)
- $\int_a^b \cos x\,dx = \sin(b) - \sin(a) = \sin x\big|_a^b$ (by **).

(b) $(|x|)' = \operatorname{sign} x$ except at $x=0$ where the function $|x|$ is not differentiable.





$$\Rightarrow \int \operatorname{sign} x\,dx = |x| + C\ .$$

This is consistent with Example on p. 102, where we had $C=-2$.