CONTINUED FRACTIONS

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As an application of the theory of limits, we will verify that

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}=\sqrt{2}. \tag{1}$$

An object in the left side of the equation is called a *continued fraction.* Equation (1) is one example of a rich theory of continuous fractions, see Wikipedia if interested.

One can rigorously define the continuous fraction in (1) as a limit of the sequence of finite fractions of the form

$$1, \quad 1+\frac{1}{2}, \quad 1+\cfrac{1}{2+\cfrac{1}{2}}, \quad 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}, \ldots$$

The first six terms of this sequence are approximately

$$1, 1.5, 1.4, 1.417, 1.4138, 1.41429,$$

so the sequence indeed seems to converge to $\sqrt{2} \approx 1.414214$ quite fast.

The following theorem is a rigorous way to state this convergence.

Theorem 1. *Let* $a_1 = 1$ *and*

$$a_{n+1} = 1 + \frac{1}{1+a_n}, \qquad n = 1, 2, \ldots. \tag{2}$$

Then

$$\lim a_n = \sqrt{2}.$$

We shall prove the theorem by the following series of results.

Lemma 2. *Assume that* $\lim a_n = a \in \mathbb{R}$ *exists. Then*

$$\lim a_n = \sqrt{2}.$$

Proof. Taking limits of both sides of (2) and using limit theorems **(do this)**, we obtain

$$a = 1 + \frac{1}{1+a}.$$

Solving this equation **(check this)** yields $a = \sqrt{2}$. □

Unfortunately (a_n) is not a monotone sequence, which makes it impossible to apply Weierstrass theorem. However, the subsequences of even terms $(a_{2n})_{n=1}^{\infty}$ and of odd terms $(a_{2n+1})_{n=1}^{\infty}$ both turn out to be monotone and bounded. This will allow us to use Weierstrass theorem separately for each subsequence, and them "glue" them together.

Lemma 3. *One has*

$$a_{2n} \geq \sqrt{2}, \qquad a_{2n+2} \leq a_{2n} \qquad \textit{for all } n.$$

Proof. A calculation gives

$$a_{2n+2} = \frac{4 + 3a_{2n}}{3 + 2a_{2n}} \quad \text{for all } n.$$

Then both inequalities in the statement of the lemma can be proved by induction. **(Do this.)** □

Lemma 3 states that the sequence $(a_{2n})_{n=1}^{\infty}$ is non-increasing and bounded below. By Weierstrass theorem, this sequence converges. Arguing similarly to Lemma 2 **(check this)** we deduce that

$$\lim a_{2n} = \sqrt{2}.$$

A similar argument **(give it)** for $(a_{2n+1})_{n=1}^{\infty}$ gives

$$\lim a_{2n+1} = \sqrt{2}.$$

Combining the two subsequences together **(how?)**, one concludes that

$$\lim a_n = \sqrt{2}.$$

This proves the Theorem. □