CONTINUED FRACTIONS

As an application of the theory of limits, we will verify that

\[ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}} = \sqrt{2}. \]

An object in the left side of the equation is called a continued fraction. Equation (1) is one example of a rich theory of continuous fractions, see Wikipedia if interested.

One can rigorously define the continuous fraction in (1) as a limit of the sequence of finite fractions of the form

\[
1, \quad 1 + \frac{1}{2}, \quad 1 + \frac{1}{2 + \frac{1}{2}}, \quad 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \ldots
\]

The first six terms of this sequence are approximately

1, 1.5, 1.4, 1.417, 1.4138, 1.41429,

so the sequence indeed seems to converge to \( \sqrt{2} \approx 1.414214 \) quite fast.

The following theorem is a rigorous way to state this convergence.

**Theorem 1.** Let \( a_1 = 1 \) and

\[
a_{n+1} = 1 + \frac{1}{1 + a_n}, \quad n = 1, 2, \ldots
\]

Then

\[ \lim_{n \to \infty} a_n = \sqrt{2}. \]

We shall prove the theorem by the following series of results.

**Lemma 2.** Assume that \( \lim a_n = a \in \mathbb{R} \) exists. Then

\[ \lim_{n \to \infty} a_n = \sqrt{2}. \]

**Proof.** Taking limits of both sides of (2) and using limit theorems (do this), we obtain

\[ a = 1 + \frac{1}{1 + a}. \]

Solving this equation (check this) yields \( a = \sqrt{2} \).

Unfortunately \( (a_n) \) is not a monotone sequence, which makes it impossible to apply Weierstrass theorem. However, the subsequences of even terms \( (a_{2n})_{n=1}^\infty \) and of odd terms \( (a_{2n+1})_{n=1}^\infty \) both turn out to be monotone and bounded. This will allow us to use Weierstrass theorem separately for each subsequence, and then “glue” them together.
Lemma 3. One has
\[ a_{2n} \geq \sqrt{2}, \quad a_{2n+2} \leq a_{2n} \quad \text{for all } n. \]

Proof. A calculation gives
\[ a_{2n+2} = \frac{4 + 3a_{2n}}{3 + 2a_{2n}} \quad \text{for all } n. \]

Then both inequalities in the statement of the lemma can be proved by induction. (Do this.)

Lemma 3 states that the sequence \( (a_{2n})_{n=1}^{\infty} \) is non-increasing and bounded below. By Weierstrass theorem, this sequence converges. Arguing similarly to Lemma 2 (check this) we deduce that
\[ \lim_{n \to \infty} a_{2n} = \sqrt{2}. \]

A similar argument (give it) for \( (a_{2n+1})_{n=1}^{\infty} \) gives
\[ \lim_{n \to \infty} a_{2n+1} = \sqrt{2}. \]

Combining the two subsequences together (how?), one concludes that
\[ \lim_{n \to \infty} a_n = \sqrt{2}. \]

This proves the Theorem. \( \square \)