1 (30 pts.) No justification is required. **No partial credit.**

Part I: Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones.

(T) If A in an $n \times n$ matrix such that $|A\vec{v}| = |\vec{v}|$ for every $\vec{v} \in \mathbb{R}^n$, then for any $\vec{b} \in \mathbb{R}^n$, the equation $A^T\vec{x} = \vec{b}$ has a unique solution.

(T) If A commutes with B, then A^T commutes with B^T.

(T) If 0 is an eigenvalue of A, then $\text{rank}(A) < n$.

(F) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

(F) If A is an orthogonal matrix, then there exists an orthonormal eigenbasis for A.

$\boxed{\textbf{Part II:}}$ In this part we suppose A is a 3×3 matrix with eigenvalues $1, 2$ and 3. Which of the following statements must be true? Put a (T) before the correct ones and an (F) before the wrong ones.

(T) There is some $\vec{v} \in \mathbb{R}^3$ so that the equation $(A - 3I)x = v$ has no solution.

(F) A is positive definite.

(T) A is invertible.

(T) A must be diagonalizable.

(T) A can not be an orthogonal matrix.

2 (30 pts.) No justification is required. **No partial credit.**

Part I: Fill in the blanks.

(a) The inverse of the matrix $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ is $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$.

(b) If A is a 5×5 matrix with $\det(A) = 3$, then $\operatorname{rank}(A) =$ 5

(c) The volume of the parallelogram defined by the vectors $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is $\sqrt{5}$

(d) If A is a 3×3 real matrix, and we know that 1 and $2 + 3i$ are two eigenvalues of A, then the trace of A is 5

(e) The dimension of the space of all matrices A that satisfies the equation $A \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} A$ is 2

$\boxed{\textbf{Part II:}}$ For each of the following problem, write down a matrix satisfying the given condition.

(f) A 2×2 non-diagonal matrix with eigenvalues 1, 2.

e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ ……

(g) A square matrix that is not diagonalizable.

e.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ……

(h) A 4×5 matrix with rank 3.

e.g. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 4 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \end{pmatrix}$ ……

(i) An orthogonal matrix all of whose entires are nonzero.

$\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ ……

(j) A 3×3 matrix with $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as one of its eigenvector.

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & -1 & 0 \\ 6 & 0 & -2 \\ 0 & 3 & -2 \end{pmatrix}$ ……

3 (15 pts.) *Makeup your own problem:* Give an example of a matrix A all of whose entries are nonzero, and a nonzero vector $\vec{b}$, so that the solutions to the equation $A\vec{x} = \vec{b}$ form a line in $\mathbb{R}^3$. Then find all solutions to your equation.

According to the problem, we know

① A is $n \times 3$ matrix

② RREF(A) contains $3-1=2$ leading 1's. $\quad 1 = \dim$ of line.

For example, we could take RREF$(A|\vec{b})$ to be

$$\text{RREF}(A|\vec{b}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

Using row transforms, we could get such an A whose entries are nonzero:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & 5 \end{array}\right)$$

So we can take $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

The solution to $A\vec{x} = \vec{b}$ is $\begin{cases} x_1 = 1 \\ x_2 = 1 - t \\ x_3 = t \end{cases}$

4 (20 pts.) Let $\mathcal{B} = \{1+x, 1+x^2, x+x^2\}$.

(a) Show that $\mathcal{B}$ is a basis of $\mathcal{P}_2$.

(b) Let $p_0(x) = 1+x+x^2$. Compute $[p_0(x)]_{\mathcal{B}}$.

(c) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation $T(p(x)) = p(x) - p'(x)$. Write down the matrix A of the transformation T with respect to the basis $\mathcal{B}$.

(d) Is T an isomorphism?

(a). If $0 = c_1(1+x) + c_2(1+x^2) + c_3(x+x^2) = (c_1+c_2) + (c_1+c_3)x + (c_2+c_3)x^2$

then $c_1+c_2=0$, $c_1+c_3=0$, $c_2+c_3=0$ $\Rightarrow$ $c_1+c_2+c_3 = \frac{1}{2}(c_1+c_2+c_1+c_3+c_2+c_3)=0$

$\Rightarrow c_1 = (c_1+c_2+c_3) - (c_2+c_3) = 0$, $c_2 = 0$, $c_3 = 0$.

So $\mathcal{B}$ is a basis of $\mathcal{P}_2$. $[\dim \mathcal{P}_2 = 3]$.

(b) $1+x+x^2 = \frac{1}{2}(1+x+1+x^2+x+x^2) \Rightarrow (1+x+x^2)_{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

(c) $T(1+x) = 1+x-1 = x = \frac{1}{2}[(1+x)+(x+x^2)-(1+x^2)]$

$T(1+x^2) = 1+x^2-2x = 2(1+x^2) - (1+x) - (x+x^2)$

$T(x+x^2) = x+x^2-1-2x = \frac{1}{2}[(x+x^2)+(1+x^2)-3(1+x)]$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{2} & -1 & -\frac{3}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

(d) $\det A = \frac{1}{4} \det \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{4} \det \begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix} = 1 \neq 0$

$\Rightarrow$ T is an isomorphism.

5 (10 pts.) Calculate the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

[No partial credit!]

$$\det A = \det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} = \det\begin{pmatrix} 4 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = 4(-1)(-1)(-1)(-1) = 4$$

[Alternatively: eigenvalues of $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ are $5, 0, 0, 0, 0$

$\Rightarrow$ eigenvalues of A are $4, -1, -1, -1, -1$

$\Rightarrow \det A = 4(-1)(-1)(-1)(-1) = 4$]

6 (20 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(a) Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A.

(c) Write down an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

(d) Compute A^{2012}.

(a) characteristic eqn $0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 & 0 \\ 3 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda)\left[(1-\lambda)(4-\lambda) - 6\right] = (5-\lambda)(\lambda^2 - 5\lambda - 2)$

$\Rightarrow \lambda_1 = 5, \quad \lambda_2 = \frac{5+\sqrt{33}}{2}, \quad \lambda_3 = \frac{5-\sqrt{33}}{2}$

(b). $E_5 = \mathrm{Ker}\begin{pmatrix} -4 & 2 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$E_{\frac{5+\sqrt{33}}{2}} = \mathrm{Ker}\begin{pmatrix} -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 & 0 \\ 3 & \frac{3}{2} - \frac{\sqrt{33}}{2} & 0 \\ 0 & 0 & \frac{5}{2} - \frac{\sqrt{33}}{2} \end{pmatrix} = \mathrm{span}\begin{pmatrix} 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \\ 0 \end{pmatrix}$

$E_{\frac{5-\sqrt{33}}{2}} = \mathrm{Ker}\begin{pmatrix} -\frac{3}{2} + \frac{\sqrt{33}}{2} & 2 & 0 \\ 3 & \frac{3}{2} + \frac{\sqrt{33}}{2} & 0 \\ 0 & 0 & \frac{5}{2} + \frac{\sqrt{33}}{2} \end{pmatrix} = \mathrm{span}\begin{pmatrix} 2 \\ \frac{3}{2} - \frac{\sqrt{33}}{2} \\ 0 \end{pmatrix}$

(c) $P = \begin{pmatrix} 0 & 2 & 2 \\ 0 & \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{5+\sqrt{33}}{2} & 0 \\ 0 & 0 & \frac{5-\sqrt{33}}{2} \end{pmatrix}$

(d) $A^{2012} = P D^{2012} P^{-1}$

$= \begin{pmatrix} 0 & 2 & 2 \\ 0 & \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5^{2012} & & \\ & \left(\frac{5+\sqrt{33}}{2}\right)^{2012} & \\ & & \left(\frac{5-\sqrt{33}}{2}\right)^{2012} \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 0 & \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \\ 1 & 0 & 0 \end{pmatrix}^{-1}$

7 (20 pts.) Consider the quadractic form

$$q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3.$$

(a) Write down the matrix A for q.

(b) Find an orthogonal matrix Q and a diagonal matrix D so that $A = Q^T D Q$.

(c) Diagonalize q.

(d) Is q positive definite?

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

(b) $0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 & 1 \\ 2 & 2-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{pmatrix} = \det\begin{pmatrix} 0 & 2+2(\lambda-1) & 1+(1-\lambda)(\lambda-1) \\ 0 & 2-\lambda-4 & 2-2(1-\lambda) \\ 1 & 2 & 1-\lambda \end{pmatrix} = \det\begin{pmatrix} 2\lambda & -\lambda^2+2\lambda \\ -\lambda-2 & 2\lambda \end{pmatrix}$

$= 4\lambda^2 - (-\lambda-2)(-\lambda^2+2\lambda) = 4\lambda^2 - (\lambda+2)\lambda(\lambda-2) = \lambda[4\lambda - \lambda^2 + 4] = \lambda(-\lambda^2+4\lambda+4)$

$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = \frac{-4+\sqrt{32}}{-2} = 2-2\sqrt{2}, \quad \lambda_3 = 2+2\sqrt{2}$

$E_0 = \mathrm{Ker}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rightsquigarrow v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$E_{2-2\sqrt{2}} = \mathrm{Ker}\begin{pmatrix} -1+2\sqrt{2} & 2 & 1 \\ 2 & 2\sqrt{2} & 2 \\ 1 & 2 & -1+2\sqrt{2} \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 0 & 4-4\sqrt{2} & -8+4\sqrt{2} \\ 0 & 2\sqrt{2}-4 & 4-4\sqrt{2} \\ 1 & 2 & -1+2\sqrt{2} \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow v_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$E_{2+2\sqrt{2}} = \mathrm{Ker}\begin{pmatrix} -1-2\sqrt{2} & 2 & 1 \\ 2 & -2\sqrt{2} & 2 \\ 1 & 2 & -1-2\sqrt{2} \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 0 & 4+4\sqrt{2} & -8-4\sqrt{2} \\ 0 & -2\sqrt{2}-4 & 4+4\sqrt{2} \\ 1 & 2 & -1-2\sqrt{2} \end{pmatrix} = \mathrm{span}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \Rightarrow v_3 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$\Rightarrow Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & \frac{-\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-2\sqrt{2} & 0 \\ 0 & 0 & 2+2\sqrt{2} \end{pmatrix}$

(c). $\vec{y} = Q\vec{x} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1 - x_3) \\ \frac{1}{2}(x_1 - 2\sqrt{2}x_2 + x_3) \\ \frac{1}{2}(x_1 + 2\sqrt{2}x_2 + x_3) \end{pmatrix}$

$q(\vec{x}) = (2-2\sqrt{2})\left(\frac{x_1 - 2\sqrt{2}x_2 + x_3}{2}\right)^2 + (2+2\sqrt{2})\left(\frac{x_1 + 4\sqrt{2}x_2 + x_3}{2}\right)^2$

(d) No.