1 (15 pts.) Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No justification is required.)

(T) If A is the matrix of an orthogonal projection, then $A^2 = A$.

(F) If A and B are both orthogonal $n \times n$ matrices, then $AB = BA$.

e.g. $\underline{n=2}$. reflections

(T) The set of all smooth functions $f(x)$ such that $\int_{-1}^{1} f(x)dx = 0$ is a linear space.

(F) The map $T(f) = \begin{pmatrix} f(0) & f(1) \\ f(2) & f(3) \end{pmatrix}$ is an isomorphism from P_4 to $\mathbb{R}^{2\times 2}$.

$\dim P_4 = 5 > \dim \mathbb{R}^{2\times 2} = 4$

(T) If the entries of two vectors $\vec{v}$ and $\vec{w}$ are all negative, then the angle between $\vec{v}$ and $\vec{w}$ must be an acute angle.

$\vec{v}\cdot\vec{w} > 0 \Rightarrow \theta < \frac{\pi}{2}$

2 (15 pts.) Fill in the blanks. (No justification is required. **No partial credit.**)

(a) The trace of the matrix $\begin{pmatrix} 3 & 4 & 5 \\ 12 & 3 & 14 \\ 0 & 0 & 1 \end{pmatrix}$ is $\underline{\quad 7 \quad}$.

(b) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be orthonormal vectors in $\mathbb{R}^{15}$. Then the length of the vector $v = \vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 - 2\vec{v}_4$ is $\underline{\quad \sqrt{10} \quad}$.

(c) The dimension of the space of "all polynomials in $\mathcal{P}_3$ that are also odd functions" is $\underline{\quad 2 \quad}$.

(d) Under the basis

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$, the coordinate vector of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$.

(e) Write down a 4×4 orthogonal matrix that is not the identity matrix

e.g. $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

3 (15 pts.) Let V be the subset of $\mathbb{R}^{3\times 3}$ consisting of all 3×3 matrices A such that $A^T = -A$.

(a) Argue that V is a linear subspace of P_3.

(b) Find a basis of V.

(c) Consider the linear transformation $T : V \to V$ defined by

$$T(A) = S^T A S,$$

where S is the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix}$. Find the matrix of T with respect to the basis you get in part (b).

(a). $A^T = -A,\ B^T = -B \Rightarrow (\alpha A + \beta B)^T = \alpha A^T + \beta B^T = -\alpha A - \beta B = -(\alpha A + \beta B)$.

(b) $A^T = -A \Rightarrow A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Basis: $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

(c) $T(A_1) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 4 \\ -2 & -4 & 0 \end{pmatrix} = A_1 + 2A_2 + 4A_3$

$\Rightarrow [T(A_1)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$T(A_2) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_1$

$\Rightarrow [T(A_2)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T(A_3) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} = A_1 + 2A_2 + 2A_3$

$\Rightarrow [T(A_3)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\mathcal{B}$-Matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 0 & 2 \end{pmatrix}$.

4 (15 pts.) (a) Find the least square solution to the system $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$.

(b) Find a linear function of the form $f(t) = c_0 + c_1 t$ that best fits the data $(0,1), (1,2), (2,3), (3,5)$.

(c) Find the matrix of the orthogonal projection onto the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$,

$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}^{-1} = \frac{1}{56-36} \begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 14 & -6 \\ -6 & 4 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix}$

$\vec{x}^* = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 11 \\ 23 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 8 \\ 13 \end{pmatrix}$

(b). Same $\overset{\text{eqn}}{/}$ as (a)!

$f(t) = \frac{8}{10} + \frac{13}{10} t$

(c) The matrix is

$A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \cdot \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

$= \frac{1}{10} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 7 & 4 & 1 & -2 \\ -3 & -1 & 1 & 3 \end{pmatrix}$

$= \frac{1}{10} \begin{pmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{pmatrix}$

5 (20 pts.) Let $V = \operatorname{Im}(A)$, where $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$.

(a) Find an orthonormal basis of V.

(b) Find the QR decomposition of A.

(c) Find the orthogonal projection of $\vec{v} = (1\ 2\ 3\ 4)^T$ onto V.

(d) Find an orthonormal basis of $V^\perp$.

Sol: (a) Same as what we did in class, we will get

$$\vec{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

(b).

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{aligned} \operatorname{Proj}_V(\vec{v}) &= (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2 + (\vec{u}_3 \cdot \vec{v})\vec{u}_3 \\ &= 5\vec{u}_1 + 2\vec{u}_2 + 0 \cdot \vec{u}_3 \\ &= \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{7}{2} \\ \frac{7}{2} \end{pmatrix}. \end{aligned}$$

(d)

$$\left.\begin{aligned} &\vec{v} - \operatorname{Proj}_V(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{7}{2} \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &\left\| \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\| = 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} &\vec{u}_4 = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \\ &\dim V^\perp = 1 \end{aligned}\right\} \Rightarrow \text{basis of } V^\perp \text{ is } \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

6 (20 pts.) Let V be an inner product space, and X_1, X_2, X_3 three elements in V. Suppose we know that $\langle X_i, X_j\rangle$ is the entry a_{ij} of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}.$$

Use this information to answer the following questions.

(a) Find $|X_1|$.

(b) Find the angle enclosed by X_2 and X_3.

(c) Find $|X_1 + X_2|$.

(d) Find $\mathrm{Proj}_V(X_3)$, where $V = \mathrm{span}(X_1, X_2)$.

For each of the following matrices B, there is no way to find elements X_1, X_2, X_3 so that the entry b_{ij} is $\langle X_i, X_j\rangle$. Explain why (case by case).

(e) $B = \begin{pmatrix} -1 & 3 & 0 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 5 & 8 \\ 0 & 8 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 3 & 1 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}$

(a) $|X_1| = \sqrt{\langle X_1, X_1\rangle} = 1$

(b) $\frac{\langle X_2, X_3\rangle}{|X_2|\,|X_3|} = \frac{8}{\sqrt{50}\sqrt{4}} = \frac{4}{\sqrt{50}} \Rightarrow \theta = \arccos\frac{4}{\sqrt{50}}$

(c) $|X_1+X_2| = \sqrt{|X_1+X_2|^2} = \sqrt{|X_1|^2 + 2\langle X_1, X_2\rangle + |X_2|^2} = \sqrt{1+2\cdot 3+50} = \sqrt{57}$

(d) Orthonormal basis of V.

$Y_1 = \frac{X_1}{|X_1|} = X_1$, $Y_2^{(1)} = X_2 - \langle Y_1, X_2\rangle Y_1 = X_2 - 3X_1$, $|Y_2^{(1)}| = \sqrt{|X_2 - 3X_1|^2}$

$= \sqrt{9 - 6\cdot 3 + 50}$

$\Rightarrow Y_2 = \frac{1}{\sqrt{41}}(X_2 - 3X_1)$ $= \sqrt{41}$

$\therefore \mathrm{Proj}_V(X_3) = \langle Y_1, X_3\rangle Y_1 + \langle Y_2, X_3\rangle Y_2 = 0\cdot Y_1 + \frac{1}{41}\langle X_2 - 3X_1, X_3\rangle (X_2 - 3X_1)$

$= \frac{1}{41}(8 - 3\cdot 0)(X_2 - 3X_1) = \frac{8}{41}(X_2 - 3X_1)$.

(e) $\langle X_1, X_1\rangle = -1 < 0$. (f) $\langle X_1, X_2\rangle = 3 > |X_1|\,|X_2| = 1\cdot\sqrt{5}$ (g) $\langle X_1, X_3\rangle = 1 \neq \langle X_3, X_1\rangle$.