1 (15 pts.) Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No justification is required.)

(T) The inverse of an orthogonal matrix is still an orthogonal matrix.

(F) If A and B are both $n \times n$ symmetric matrices, so is AB.

e.g. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

(F) The set of all polynomials $p(x)$ of degree no more than 5 such that $p'(0) \neq 0$ is a linear space.

0 is not in this set.

(T) The linear transformation $T(A) = A \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ is an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

$T^{-1}(A) = A \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1}$.

(T) For any 2×2 matrix A, $\det(A^T) = \det(A)$.

$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2 (15 pts.) Fill in the blanks. (No justification is required. **No partial credit.**)

(a) The trace of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 8 & 7 & 6 \end{pmatrix}$ is ___10___.

(b) The length of the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ is ___3___.

(c) The dimension of the space of all symmetric 3×3 matrices is ___6___.

$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$

(d) Suppose $A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ b & c & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is an orthogonal matrix, then the values of the missing entries are $a =$ ___$-\frac{2}{\sqrt{6}}$___, $b =$ ___0___, $c =$ ___$\frac{1}{\sqrt{2}}$___.

$\boxed{\text{row 1} \perp \text{row 2}}$

$\frac{1}{\sqrt{3}}a + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{6}} = 0 \Rightarrow a = -\frac{2}{\sqrt{6}}$

$\boxed{\text{column 2} \perp \text{column 3}}$

$\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}} + c \cdot \left(-\frac{1}{\sqrt{2}}\right) = 0 \Rightarrow c = \frac{1}{\sqrt{2}}$

$\boxed{\text{row 3 length} = 1}$

$b^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1 \Rightarrow b = 0$

(e) Under the basis $\mathcal{B} = (1, x-1, x^2-1, x^3-1, x^4-1)$ of $\mathcal{P}_4$, the coordinate vector of the polynomial $1 + x + x^2 + x^3 + x^4$ is $\begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

$1 + x + x^2 + x^3 + x^4$

$= 5 + (x-1) + (x^2-1) + (x^3-1) + (x^4-1)$

3 (15 pts.) Let V be the subset of $\mathcal{P}_3$ consisting of all polynomials $p(x)$ of degree no more than 3 that satisfy the relation $p(1) = p(-1)$.

(a) Argue that V is a linear subspace of $\mathcal{P}_3$.

(b) Find a basis of V.

(c) Consider the linear transformation $T : V \to V$ defined by

$$T(p(x)) = p(-x).$$

Find the matrix of T with respect to the basis you get in part (b).

(a) If $p(x) = ax^3 + bx^2 + cx + d \in V$, $\tilde{p}(x) = \tilde{a}x^3 + \tilde{b}x^2 + \tilde{c}x + \tilde{d} \in V$ $\Rightarrow$ $\alpha p(x) + \beta \tilde{p}(x) = (\alpha a + \beta \tilde{a})x^3 + (\alpha b + \beta \tilde{b})x^2 + (\alpha c + \beta \tilde{c})x + (\alpha d + \beta \tilde{d})$ degree no more than 3

$\alpha p(1) + \beta \tilde{p}(1) = \alpha p(-1) + \beta \tilde{p}(-1)$

$\Rightarrow \alpha p(x) + \beta \tilde{p}(x) \in V.$

(b) $\left.\begin{array}{l} p(x) = ax^3 + bx^2 + cx + d \\ p(1) = p(-1) \end{array}\right\} \Rightarrow a + b + c + d = -a + b - c + d \Rightarrow a + c = 0 \Rightarrow c = -a.$

$\therefore p(x) = ax^3 + bx^2 - ax + d = a(x^3 - x) + bx^2 + d \cdot 1$

Basis: $x^3 - x,\ x^2,\ 1.$

(c). $T(x^3 - x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) \Rightarrow [T(x^3 - x)]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$T(x^2) = (-x)^2 = x^2 \Rightarrow [T(x^2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$T(1) = 1 \Rightarrow [T(1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ Matrix of T is

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4 (15 pts.) Consider the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ in $\mathbb{R}^4$. Suppose we know that $\vec{v}_i \cdot \vec{v}_j$ is the entry a_{ij} of the matrix $A = \begin{pmatrix} 10 & 10 & 3 \\ 10 & 19 & 3 \\ 3 & 3 & 2 \end{pmatrix}$. Use this information to answer the following questions.

(a) Find $|\vec{v}_2|$.

(b) Find the cosine of the angle enclosed by $\vec{v}_1$ and $\vec{v}_3$.

(c) Find $|\vec{v}_1 - \vec{v}_2|$.

(d) Find $\mathrm{Proj}_V(\vec{v}_3)$, where $V = \mathrm{span}(\vec{v}_1, \vec{v}_2)$.

(e) Find an orthonormal basis of $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. The vectors should be expressed as linear combinations of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

(a) $|\vec{v}_2| = \sqrt{\vec{v}_2 \cdot \vec{v}_2} = \sqrt{19}$

(b) $\cos\theta = \dfrac{\vec{v}_1 \cdot \vec{v}_3}{|\vec{v}_1| \cdot |\vec{v}_3|} = \dfrac{\vec{v}_1 \cdot \vec{v}_3}{\sqrt{\vec{v}_1 \cdot \vec{v}_1}\sqrt{\vec{v}_3 \cdot \vec{v}_3}} = \dfrac{3}{\sqrt{10}\sqrt{2}} = \dfrac{3}{\sqrt{20}}$

(c) $|\vec{v}_1 - \vec{v}_2| = \sqrt{(\vec{v}_1 - \vec{v}_2)\cdot(\vec{v}_1 - \vec{v}_2)} = \sqrt{\vec{v}_1\cdot\vec{v}_1 - \vec{v}_1\cdot\vec{v}_2 - \vec{v}_2\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2} = \sqrt{10-10-10+19} = \sqrt{9} = 3$.

(d) $\vec{u}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|} = \dfrac{1}{\sqrt{10}}\vec{v}_1$

$\vec{v}_2^{(1)} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \vec{v}_2 - \frac{1}{\sqrt{10}}(\vec{v}_1 \cdot \vec{v}_2)\frac{1}{\sqrt{10}}\vec{v}_1 = \vec{v}_2 - \frac{1}{10}10\vec{v}_1 = \vec{v}_2 - \vec{v}_1 \Big| \Rightarrow \vec{u}_2 = \frac{1}{3}(\vec{v}_2 - \vec{v}_1)$

$|\vec{v}_2^{(1)}| = |\vec{v}_2 - \vec{v}_1| = |\vec{v}_1 - \vec{v}_2| = 3$

$\mathrm{Proj}_V(\vec{v}_3) = (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 = \frac{1}{10}(\underbrace{\vec{v}_1 \cdot \vec{v}_3}_{3})\vec{v}_1 + \frac{1}{9}(\underbrace{\vec{v}_2 \cdot \vec{v}_3}_{3} - \underbrace{\vec{v}_1 \cdot \vec{v}_3}_{3})(\vec{v}_2 - \vec{v}_1) = \frac{3}{10}\vec{v}_1$

(e) We already have $\vec{u}_1 = \frac{1}{\sqrt{10}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{3}(\vec{v}_2 - \vec{v}_1)$

$\vec{v}_3^{(1)} = \vec{v}_3 - \mathrm{Proj}_V(\vec{v}_3) = \vec{v}_3 - \frac{3}{10}\vec{v}_1$

$|\vec{v}_3^{(1)}| = \sqrt{(\vec{v}_3 - \frac{3}{10}\vec{v}_1)\cdot(\vec{v}_3 - \frac{3}{10}\vec{v}_1)} = \sqrt{\vec{v}_3\cdot\vec{v}_3 - \frac{3}{10}\vec{v}_3\cdot\vec{v}_1 - \frac{3}{10}\vec{v}_1\cdot\vec{v}_3 + \frac{9}{100}\vec{v}_1\cdot\vec{v}_1}$

$= \sqrt{2 - \frac{3}{10}3 - \frac{3}{10}\cdot 3 + \frac{9}{100}\cdot 10} = \sqrt{2 - \frac{18}{10} + \frac{9}{10}} = \sqrt{\frac{11}{10}}$

$\therefore \vec{u}_3 = \dfrac{1}{\sqrt{\frac{11}{10}}}(\vec{v}_3 - \frac{3}{10}\vec{v}_1)$.

5 (20 pts.) Let $V = \mathrm{Im}(A)$, where $A = \begin{pmatrix}2&1&1\\0&1&2\\1&1&1\\2&0&0\end{pmatrix}$.

(a) Find an orthonormal basis of V.

(b) Find the QR decomposition of A.

(c) Find the matrix of the orthogonal projection map Proj_V.

(d) Find an orthonormal basis of $V^\perp$.

(a) $\vec{v}_1 = \begin{pmatrix}2\\0\\1\\2\end{pmatrix}$, $|\vec{v}_1| = 3 \Rightarrow \vec{u}_1 = \frac{1}{3}\begin{pmatrix}2\\0\\1\\2\end{pmatrix}$, $\vec{u}_1\cdot\vec{v}_2 = 1$, $\vec{u}_1\cdot\vec{v}_3 = 1$

$\vec{v}_2^{(\perp)} = \vec{v}_2 - (\vec{v}_2\cdot\vec{u}_1)\vec{u}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\0\\1\\2\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\1\\\frac{2}{3}\\-\frac{2}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\3\\2\\-2\end{pmatrix}$, $|\vec{v}_2^{(\perp)}| = \frac{1}{3}\sqrt{1+9+4+4} = \sqrt{2}$

$\Rightarrow \vec{u}_2 = \frac{1}{3\sqrt{2}}\begin{pmatrix}1\\3\\2\\-2\end{pmatrix}$, $\vec{u}_2\cdot\vec{v}_3 = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}}$

$\vec{v}_3^{(\perp)} = \vec{v}_3 - (\vec{u}_1\cdot\vec{v}_3)\vec{u}_1 - (\vec{u}_2\cdot\vec{v}_3)\vec{u}_2 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\0\\1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\3\\2\\-2\end{pmatrix} = \begin{pmatrix}-\frac{1}{6}\\\frac{1}{2}\\-\frac{1}{3}\\\frac{1}{3}\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-1\\3\\-2\\2\end{pmatrix}$

$|\vec{v}_3^{(\perp)}| = \frac{1}{6}\sqrt{1+9+4+4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \vec{u}_3 = \frac{\sqrt{2}}{6}\begin{pmatrix}-1\\3\\-2\\2\end{pmatrix}$

O.N. basis: $\mathcal{B} = \frac{1}{3}\begin{pmatrix}2\\0\\1\\2\end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix}1\\3\\2\\-2\end{pmatrix}, \frac{\sqrt{2}}{6}\begin{pmatrix}-1\\3\\-2\\2\end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix}-1\\3\\-2\\2\end{pmatrix}$.

(b). QR:

$$A = \begin{pmatrix}\frac{2}{3} & \frac{1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}}\\ 0 & \frac{3}{3\sqrt{2}} & \frac{3}{3\sqrt{2}}\\ \frac{1}{3} & \frac{2}{3\sqrt{2}} & \frac{-2}{3\sqrt{2}}\\ \frac{2}{3} & \frac{-2}{3\sqrt{2}} & \frac{2}{3\sqrt{2}}\end{pmatrix}\begin{pmatrix}3 & 1 & 1\\ 0 & \sqrt{2} & \frac{3}{\sqrt{2}}\\ 0 & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

(c) Matrix of $\mathrm{Proj}_V = QQ^T = \frac{1}{3\sqrt{2}}\frac{1}{3\sqrt{2}}\begin{pmatrix}2\sqrt{2} & 1 & -1\\ 0 & 3 & 3\\ \sqrt{2} & 2 & -2\\ 2\sqrt{2} & -2 & 2\end{pmatrix}\begin{pmatrix}2\sqrt{2} & 0 & \sqrt{2} & 2\sqrt{2}\\ 1 & 3 & 2 & -2\\ -1 & 3 & -2 & 2\end{pmatrix} = \frac{1}{18}\begin{pmatrix}10 & 0 & 8 & 4\\ 0 & 18 & 0 & 0\\ 8 & 0 & 10 & -4\\ 4 & 0 & -4 & 16\end{pmatrix}$

$= \begin{pmatrix}\frac{5}{9} & 0 & \frac{4}{9} & \frac{2}{9}\\ 0 & 1 & 0 & 0\\ \frac{4}{9} & 0 & \frac{5}{9} & \frac{-2}{9}\\ \frac{2}{9} & 0 & \frac{-2}{9} & \frac{8}{9}\end{pmatrix}$

(d) ($\dim V^\perp = 4 - \dim V = 4 - 3 = 1$)

$V^\perp = (\mathrm{Im}A)^\perp = \ker(A^T)$

RREF: $\begin{pmatrix}2&0&1&2\\1&1&1&0\\1&2&1&0\end{pmatrix} \to \begin{pmatrix}1&1&1&0\\1&2&1&0\\2&0&1&2\end{pmatrix} \to \begin{pmatrix}1&1&1&0\\0&1&0&0\\0&-2&-1&2\end{pmatrix} \to \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&-1&2\end{pmatrix} \to \begin{pmatrix}1&0&0&2\\0&1&0&0\\0&0&1&-2\end{pmatrix}$

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$\Rightarrow \vec{x} = t\begin{pmatrix}-2\\0\\2\\1\end{pmatrix}$. $\left|\begin{pmatrix}-2\\0\\2\\1\end{pmatrix}\right| = 3. \Rightarrow \boxed{\vec{u}_4 = \frac{1}{3}\begin{pmatrix}-2\\0\\2\\1\end{pmatrix}}$

6 (20 pts.) We define an inner product structure on $\mathbb{R}^{2\times 2}$ by the formula

$$\langle A, B\rangle = \text{Trace}\left(A^T \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} B\right).$$

(a) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, calculate $\langle A, A\rangle$.

(b) Use part (a) to check that $\langle A, A\rangle > 0$ for $A \neq 0$.

(c) Find the angle between the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(d) Find an orthonormal basis of the subspace V of $\mathbb{R}^{2\times 2}$ that contains all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

(a) $Tr\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = Tr\left(\begin{pmatrix} a+c & a+2c \\ b+d & b+2d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = Tr\begin{pmatrix} \overset{(a+c)^2+c^2}{\overset{\parallel}{a^2+ac+ac+2c^2}} & * \\ * & \underset{(b+d)^2+d^2}{\underset{\parallel}{b^2+bd+bd+2d^2}} \end{pmatrix}$

$= (a+c)^2 + (b+d)^2 + c^2 + d^2$

(b) If $A \neq 0$, then $(a+c)^2 + (b+d)^2 + c^2 + d^2 > 0$, i.e. $\langle A, A\rangle > 0$.

(c) $\left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \sqrt{(1+0)^2 + (0+1)^2 + 0^2 + 1^2} = \sqrt{3}$

$\left|\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right| = \sqrt{(0+(-1))^2 + (1+0)^2 + (-1)^2 + 0^2} = \sqrt{3}$

$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = Tr\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = Tr\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = Tr\begin{pmatrix} -1 & * \\ * & 1 \end{pmatrix} = 0.$

$\Rightarrow \theta = \frac{\pi}{2}$.

(d). A basis of V: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we get an o.n. basis $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.