MATH 419 SECTION 001

FINAL

December 20, 2012, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- Theorems listed at the end can be used without proof.
- Both sides of a US letter size paper ($8.5'' \times 11''$) with notes is OK.
- **NO CALCULATOR, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	10	
2	14	
3	12	
4	6	
5	6	
6	12	
7	20	
TOTAL	80	

Problem 1. (10 points) Find the rank of the matrices.

$$\text{(a)}\quad \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(b)}\quad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}, \qquad \text{(c)}\quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

Solution (a) [3] 3, (b) [3] 1, (c) [4] 2.

Problem 2. (14 points)

(a) Compute A^6 for $A = \begin{bmatrix} 5 & -3\sqrt{2} \\ 3\sqrt{2} & -4 \end{bmatrix}$.

(b) Compute $A^4 - 10A^3 + 21A^2 + 21A + 4$ for $A = \begin{bmatrix} 1.7 & 7.61 \\ 1 & 3.3 \end{bmatrix}$.

Solution (a) [7] Note that

$$A \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = 2 \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = - \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}.$$

We have

$$A^6 = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2^6 & 0 \\ 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 127 & -63\sqrt{2} \\ 63\sqrt{2} & -62 \end{bmatrix}.$$

(b) [7] By the Cayley-Hamilton theorem, $A^2 - (\operatorname{tr} A)A + \det A = A^2 - 5A - 2I_2 = 0$. Thus,

$$A^4 - 10A^3 + 21A^2 + 21A + 4 = (A^2 - 5A - 2I_n)^2 + A = A = \begin{bmatrix} 1.7 & 7.61 \\ 1 & 3.3 \end{bmatrix}.$$

(continued)

Problem 3. (12 points) Let V be the linear space spanned by e^x and e^{-x}, with the bases $\mathcal{A} = (e^x, e^{-x})$ and $\mathcal{B} = (e^x + e^{-x}, e^x - e^{-x})$. Consider the linear transformation $T(f) = f'$ from V to V.

(a) Find the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$.

(b) Find the $\mathcal{A}$-matrix A of T or the $\mathcal{B}$-matrix B of T (Choose one of them).

Solution (a) [6] The change of basis matrix satisfies $[f]_{\mathcal{A}} = S_{\mathcal{B}\to\mathcal{A}}[f]_{\mathcal{B}}$. We obtain

$$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} [e^x + e^{-x}]_{\mathcal{A}} & [e^x - e^{-x}]_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(b) [6] Note that $T(e^x) = e^x$ and $T(e^{-x}) = -e^{-x}$. Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $T(e^x + e^{-x}) = e^x - e^{-x}$ and $T(e^x - e^{-x}) = e^x + e^{-x}$. Thus,

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrices A and B are related as $B = S_{\mathcal{B}\to\mathcal{A}}^{-1} A S_{\mathcal{B}\to\mathcal{A}}$.

(continued)

Problem 4. (6 points) Find the determinant of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$.

Solution The determinant is 1.

Problem 5. (6 points) Express $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$.

Solution

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \left(\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\right) \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right).
\end{aligned}$$

Therefore,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Problem 6. (12 points) Decide if the matrix is diagonalizable or not. If possible, find an invertible S and a diagonal D such that $S^{-1}AS = D$, where A is the matrix in question.

$$\text{(a)} \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution (a) [6] Not diagonalizable. Note that $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) [6] Diagonalizable. $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that the eigenvalues are obtained as $2, 1$, and $E_2 = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$, $E_1 = \text{span}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix})$.

Alternative solution

Note that there are other correct solutions. For example, $S = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(continued)

Problem 7. (20 points) Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 5\sqrt{3} & 0 \\ 0 & 1 \\ -5 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \end{bmatrix}.$$

(a) Find the least-squares solution $\vec{x}^*$.

(b) Find the singular values of A.

(c) Find the regularized solution $\vec{x}^*_{\text{reg}}$ by taking only the largest singular value into account.

Solution (a) [4]

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} \frac{\sqrt{3}}{20} & 0 & -\frac{1}{20} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}.$$

(b) [4] By solving

$$\det\left(A^TA - \lambda I_2\right) = \begin{vmatrix} 100-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 100)(\lambda - 1) = 0,$$

We obtain $\lambda = 100, 1$. Therefore singular values σ_1 and σ_2 are obtained as

$$\sigma_1 = 10, \quad \sigma_2 = 1.$$

(c) [12] All singular values contribute in the Tikhonov regularization. So, we use the truncated SVD. Note that $(A^TA)\vec{v}_1 = \sigma_1^2\vec{v}_1$ and $(A^TA)\vec{v}_2 = \sigma_2^2\vec{v}_2$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. Then, $\vec{u}_1 = (1/\sigma_1)A\vec{v}_1 = \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ -1/2 \end{bmatrix}$. We obtain

$$\vec{x}^*_{\text{reg}} = \frac{1}{\sigma_1}\vec{v}_1(\vec{u}_1 \cdot \vec{b}) = \begin{bmatrix} 1/5 \\ 0 \end{bmatrix}.$$

Alternative solution

By the singular value decomposition, A is written as

$$A = U\Sigma V^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T.$$

Note that $(A^TA)^{-1}A^T = V\left(\Sigma^T\Sigma\right)^{-1}\Sigma^TU^T$ and $\left(\Sigma^T\Sigma\right)^{-1} = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 1 \end{bmatrix}$. We obtain

$$\vec{x}^*_{\text{reg}} = V\begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 0 \end{bmatrix}\Sigma^TU^T\vec{b} = \begin{bmatrix} \frac{\sqrt{3}}{20} & 0 & -\frac{1}{20} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}.$$

(continued)

Theorem 1. If $\ker(A) = \{\vec{0}\}$, then the linear system $A\vec{x} = \vec{b}$ has the unique least-squares solution $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$.

Theorem 2. The determinant of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

Theorem 3. Elementary row operations for determinants: (i) If B is obtained from A by dividing a row of A by a scalar k, then $\det B = (1/k) \det A$. (ii) If B is obtained from A by a row swap, then $\det B = -\det A$. (iii) If B is obtained from A by adding a multiple of a row of A to another row, then $\det B = \det A$.

Theorem 4. Let $\det A_{ij}$ be minors of an $n \times n$ matrix A whose entries are a_{ij}. Then the Laplace expansion down the jth column is $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$, and the Laplace expansion along the ith row is $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$.

Theorem 5. Consider $A\vec{x} = \vec{b}$, where A is an invertible $n \times n$ matrix. The components x_i of $\vec{x}$ are $x_i = \det(A_{\vec{b},i}) / \det A$, where $A_{\vec{b},i}$ is the matrix obtained by replacing the ith column of A by $\vec{b}$.

Theorem 6. Every $n \times n$ matrix A satisfies its own characteristic equation: $f_A(A) = 0$.

Theorem 7. (i) Matrix A is diagonalizable if and only if there exists an eigenbasis for A. (ii) If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Theorem 8. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Theorem 9. A matrix A is orthogonally diagonalizable if and only if A is symmetric. A complex matrix A is diagonalizable with a unitary matrix U if and only if A is selfadjoint.

Theorem 10. A symmetric matrix A is positive definite (positive semidefinite) if and only if all of its eigenvalues are positive (nonnegative).

Theorem 11. Any $n \times m$ matrix A can be written as $A = U\Sigma V^T$, where U is an orthogonal $n \times n$ matrix, V is an orthogonal $m \times m$ matrix, and Σ is an $n \times m$ matrix whose first r diagonal entries are the nonzero singular values $\sigma_1, \dots, \sigma_r$ of A, and all other entries are zero ($r = \mathrm{rank}(A)$). The matrix A can also be written as $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$, where $\vec{u}_i$ and $\vec{v}_i$ are the columns of U and V.

Theorem 12. Consider the linear system $A\vec{x} = \vec{b}$, where $n \times m$ $(n > m)$ matrix A is written as $A = U\Sigma V^T$. Here, $U = [\vec{u}_1 \cdots \vec{u}_n]$, $V = [\vec{v}_1 \cdots \vec{v}_m]$, and the diagonal entries of Σ are $\sigma_1, \dots, \sigma_m$. In the Tikhonov regularization with the Tikhonov regularization parameter $\alpha > 0$, the regularized solution is obtained as $\vec{x}^*_{\mathrm{reg}} = \frac{\sigma_1^2}{\sigma_1^2+\alpha^2}\frac{1}{\sigma_1}\vec{v}_1(\vec{u}_1 \cdot \vec{b}) + \cdots + \frac{\sigma_m^2}{\sigma_m^2+\alpha^2}\frac{1}{\sigma_m}\vec{v}_m(\vec{u}_m \cdot \vec{b})$. The vector $\vec{x}^*_{\mathrm{reg}}$ minimizes $\varepsilon(\vec{x}) = ||A\vec{x} - \vec{b}||^2 + ||\alpha\vec{x}||^2$. In the truncated SVD with the regularization parameter $\alpha > 0$, the regularized solution is obtained as $\vec{x}^*_{\mathrm{reg}} = \theta(\sigma_1 - \alpha)\frac{1}{\sigma_1}\vec{v}_1(\vec{u}_1 \cdot \vec{b}) + \cdots + \theta(\sigma_m - \alpha)\frac{1}{\sigma_m}\vec{v}_m(\vec{u}_m \cdot \vec{b})$, where $\theta(\cdot)$ is the step function ($\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \le 0$).