MATH 419 SECTION 001

FINAL

December 20, 2012, Instructor: Manabu Machida

Name: ____________________________________________

• To receive full credit you must show all your work.
• Theorems listed at the end can be used without proof.
• Both sides of a US letter size paper (8.5" × 11") with notes is OK.
• NO CALCULATOR, BOOKS, or OTHER NOTES.

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Problem 1. (10 points) Find the rank of the matrices.

(a) \[
\begin{bmatrix}
1 & 2 & 0 & 2 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad
(b) \ [ 0 \ 1 \ 2 \ 3 \ 4 ], \quad
(c) \ [ 1 \ 4 \ 7 ]
\]

Solution. (a) \([3]\) 3, (b) \([3]\) 1, (c) \([4]\) 2.
Problem 2. (14 points)

(a) Compute $A^6$ for $A = \begin{bmatrix} 5 & -3\sqrt{2} \\ 3\sqrt{2} & -4 \end{bmatrix}$.

(b) Compute $A^4 - 10A^3 + 21A^2 + 21A + 4$ for $A = \begin{bmatrix} 1.7 & 7.61 \\ 1 & 3.3 \end{bmatrix}$.

Solution (a) [7] Note that
\[ A \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = 2 \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = -\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}. \]

We have
\[ A^6 = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 2^6 & 0 \\ 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} 127 & -63\sqrt{2} \\ 63\sqrt{2} & -62 \end{bmatrix}. \]

(b) [7] By the Cayley-Hamilton theorem, $A^2 - (\text{tr } A)A + \det A = A^2 - 5A - 2I_2 = 0$. Thus,
\[ A^4 - 10A^3 + 21A^2 + 21A + 4 = (A^2 - 5A - 2I_n)^2 + A = A = \begin{bmatrix} 1.7 & 7.61 \\ 1 & 3.3 \end{bmatrix}. \]
Problem 3. (12 points) Let $V$ be the linear space spanned by $e^x$ and $e^{-x}$, with the bases $A = (e^x, e^{-x})$ and $B = (e^x + e^{-x}, e^x - e^{-x})$. Consider the linear transformation $T(f) = f'$ from $V$ to $V$.

(a) Find the change of basis matrix $S_{B \rightarrow A}$.

(b) Find the $A$-matrix $A$ of $T$ or the $B$-matrix $B$ of $T$ (Choose one of them).

Solution  

(a) [6] The change of basis matrix satisfies $[f]_A = S_{B \rightarrow A}[f]_B$. We obtain

$$S_{B \rightarrow A} = \begin{bmatrix} [e^x + e^{-x}]_A & [e^x - e^{-x}]_A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$ 

(b) [6] Note that $T(e^x) = e^x$ and $T(e^{-x}) = -e^{-x}$. Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Note that $T(e^x + e^{-x}) = e^x - e^{-x}$ and $T(e^x - e^{-x}) = e^x + e^{-x}$. Thus,

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

The matrices $A$ and $B$ are related as $B = S_{B \rightarrow A}^{-1}AS_{B \rightarrow A}$. 
Problem 4. (6 points) Find the determinant of the matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 6 & 10 \\
1 & 4 & 10 & 20
\end{bmatrix}.
\]

Solution  The determinant is 1.
Problem 5. (6 points) Express $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos \theta$ and $\sin \theta$.

Solution

$$\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$$
$$= (\cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3)$$
$$= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i (3 \cos^2 \theta \sin \theta - \sin^3 \theta) .$$

Therefore,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta, \quad \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$
Problem 6. (12 points) Decide if the matrix is diagonalizable or not. If possible, find an invertible $S$ and a diagonal $D$ such that $S^{-1}AS = D$, where $A$ is the matrix in question.

(a) $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$,  \hspace{1cm} (b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution  \hspace{1cm} (a) [6] Not diagonalizable. Note that $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) [6] Diagonalizable. $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that the eigenvalues are obtained as 2, 1, and $E_2 = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$, $E_1 = \text{span}(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix})$.

Alternative solution

Note that there are other correct solutions. For example, $S = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
(continued)
Problem 7. (20 points) Consider the linear system $A\vec{x} = \vec{b}$, where
\[
A = \begin{bmatrix}
5\sqrt{3} & 0 \\
0 & 1 \\
-5 & 0
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}
\sqrt{3} \\
1 \\
-1
\end{bmatrix}.
\]

(a) Find the least-squares solution $\vec{x}^\ast$.

(b) Find the singular values of $A$.

(c) Find the regularized solution $\vec{x}_{\text{reg}}^\ast$ by taking only the largest singular value into account.

Solution

(a) \[4\]
\[
\vec{x}^\ast = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix}
\frac{\sqrt{3}}{20} & 0 & -\frac{1}{20} \\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
\sqrt{3} \\
1 \\
-1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{5} \\
1 \\
0
\end{bmatrix}.
\]

(b) \[4\] By solving
\[
\det (A^T A - \lambda I_2) = \begin{vmatrix}
100 - \lambda & 0 \\
0 & 1 - \lambda
\end{vmatrix} = (\lambda - 100)(\lambda - 1) = 0,
\]
We obtain $\lambda = 100, 1$. Therefore singular values $\sigma_1$ and $\sigma_2$ are obtained as
\[
\sigma_1 = 10, \quad \sigma_2 = 1.
\]

(c) \[12\] All singular values contribute in the Tikhonov regularization. So, we use the truncated SVD. Note that $(A^T A) \vec{v}_1 = \sigma_1^2 \vec{v}_1$ and $(A^T A) \vec{v}_2 = \sigma_2^2 \vec{v}_2$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively.

Then, $\vec{u}_1 = (1/\sigma_1) A \vec{v}_1 = \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ -1/2 \end{bmatrix}$. We obtain
\[
\vec{x}_{\text{reg}}^\ast = \frac{1}{\sigma_1} \vec{v}_1 (\vec{u}_1 \cdot \vec{b}) = \begin{bmatrix}
1/5 \\
1 \\
0
\end{bmatrix}.
\]

Alternative solution

By the singular value decomposition, $A$ is written as
\[
A = U \Sigma V^T = \begin{bmatrix}
\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\
0 & 1 & 0 \\
-\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
10 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}^T.
\]

Note that $(A^T A)^{-1} A^T = V (\Sigma^T \Sigma)^{-1} \Sigma^T U^T$ and $(\Sigma^T \Sigma)^{-1} = \begin{bmatrix} \frac{1}{100} & 0 \\ 0 & 1 \end{bmatrix}$. We obtain
\[
\vec{x}_{\text{reg}}^\ast = V \begin{bmatrix}
\frac{1}{100} & 0 \\
0 & 0
\end{bmatrix} \Sigma^T U^T \vec{b} = \begin{bmatrix}
\frac{\sqrt{3}}{20} & 0 & -\frac{1}{20} \\
0 & 0 & \frac{\sqrt{3}}{20}
\end{bmatrix}
\begin{bmatrix}
\sqrt{3} \\
1 \\
-1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{5} \\
0
\end{bmatrix}.
\]
(continued)
**Theorem 1.** If \( \ker(A) = \{ \vec{0} \} \), then the linear system \( A\vec{x} = \vec{b} \) has the unique least-squares solution \( \vec{x}^* = (A^TA)^{-1}A^T\vec{b} \).

**Theorem 2.** The determinant of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

**Theorem 3.** Elementary row operations for determinants: (i) If \( B \) is obtained from \( A \) by dividing a row of \( A \) by a scalar \( k \), then \( \det B = (1/k) \det A \). (ii) If \( B \) is obtained from \( A \) by a row swap, then \( \det B = - \det A \). (iii) If \( B \) is obtained from \( A \) by adding a multiple of a row of \( A \) to another row, then \( \det B = \det A \).

**Theorem 4.** Let \( \det A_{ij} \) be minors of an \( n \times n \) matrix \( A \) whose entries are \( a_{ij} \). Then the Laplace expansion down the \( j \)th column is \( \det A = \sum_{i=1}^{n}(-1)^{i+j}a_{ij} \det(A_{ij}) \), and the Laplace expansion along the \( i \)th row is \( \det A = \sum_{j=1}^{n}(-1)^{i+j}a_{ij} \det(A_{ij}) \).

**Theorem 5.** Consider \( A\vec{x} = \vec{b} \), where \( A \) is an invertible \( n \times n \) matrix. The components \( x_i \) of \( \vec{x} \) are \( x_i = (A^{-1}\vec{b})_i = \det(A_{b,i})/\det A \), where \( A_{b,i} \) is the matrix obtained by replacing the \( i \)th column of \( A \) by \( \vec{b} \).

**Theorem 6.** Every \( n \times n \) matrix \( A \) satisfies its own characteristic equation: \( f_A(A) = 0 \).

**Theorem 7.** (i) Matrix \( A \) is diagonalizable if and only if there exists an eigenbasis for \( A \). (ii) If an \( n \times n \) matrix \( A \) has \( n \) distinct eigenvalues, then \( A \) is diagonalizable.

**Theorem 8.** \( (\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \). Euler’s formula: \( e^{i\theta} = \cos \theta + i \sin \theta \).

**Theorem 9.** A matrix \( A \) is orthogonally diagonalizable if and only if \( A \) is symmetric. A complex matrix \( A \) is diagonalizable with a unitary matrix \( U \) if and only if \( A \) is self-adjoint.

**Theorem 10.** A symmetric matrix \( A \) is positive definite (positive semidefinite) if and only if all of its eigenvalues are positive (nonnegative).

**Theorem 11.** Any \( n \times m \) matrix \( A \) can be written as \( A = U\Sigma V^T \), where \( U \) is an orthogonal \( n \times n \) matrix, \( V \) is an orthogonal \( m \times m \) matrix, and \( \Sigma \) is an \( n \times m \) matrix whose first \( r \) diagonal entries are the nonzero singular values \( \sigma_1, \ldots, \sigma_r \) of \( A \), and all other entries are zero (\( r = \text{rank}(A) \)). The matrix \( A \) can also be written as \( A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T \), where \( \vec{u}_i \) and \( \vec{v}_i \) are the columns of \( U \) and \( V \).

**Theorem 12.** Consider the linear system \( A\vec{x} = \vec{b} \), where \( n \times m \) (\( n > m \)) matrix \( A \) is written as \( A = U\Sigma V^T \). Here, \( U = [\vec{u}_1 \ldots \vec{u}_n] \), \( V = [\vec{v}_1 \ldots \vec{v}_m] \), and the diagonal entries of \( \Sigma \) are \( \sigma_1, \ldots, \sigma_m \). In the Tikhonov regularization with the Tikhonov regularization parameter \( \alpha > 0 \), the regularized solution is obtained as \( \vec{x}_{\text{reg}} = \frac{\sigma_1^2}{\sigma_1^2 + \alpha^2} \vec{v}_1 (\vec{u}_1 \cdot \vec{b}) + \cdots + \frac{\sigma_m^2}{\sigma_m^2 + \alpha^2} \vec{v}_m (\vec{u}_m \cdot \vec{b}) \). The vector \( \vec{x}_{\text{reg}} \) minimizes \( \varepsilon(\vec{x}) = ||A\vec{x} - \vec{b}||^2 + ||\alpha \vec{x}||^2 \). In the truncated SVD with the regularization parameter \( \alpha > 0 \), the regularized solution is obtained as \( \vec{x}_{\text{reg}} = \theta(\sigma_1 - \alpha) \frac{1}{\sigma_1} \vec{v}_1 (\vec{u}_1 \cdot \vec{b}) + \cdots + \theta(\sigma_m - \alpha) \frac{1}{\sigma_m} \vec{v}_m (\vec{u}_m \cdot \vec{b}) \), where \( \theta(\cdot) \) is the step function (\( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x \leq 0 \)).