Homework 10 Solutions

(10)

① $A\vec{x} = \vec{b}$, $A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

(a)

$\det A = -2$

$\det A_1 = \det \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix} = -4$

$\det A_2 = \det \begin{bmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & 3 & 1 \end{bmatrix} = -6$.

$\det A_3 = \det \begin{bmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{bmatrix} = -8$.

$x_1 = \dfrac{\det A_1}{\det A} = 2$, $x_2 = \dfrac{\det A_2}{\det A} = 3$, $x_3 = \dfrac{\det A_3}{\det A} = 4$. Unique.

(b) $\det A = 9 \neq 0 \Rightarrow$ according to Cramer's rule (or just ~~be~~ since A is invertible), the solution is unique.

One solution is $x_1 = x_2 = x_3 = 0$. $\Rightarrow$ it is unique

② $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix}$ $A_{11} = \det \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} = 0$, $A_{12} = -\det \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 0$, $A_{13} = \det \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} = 8$

$A_{21} = -\det \begin{bmatrix} 0 & 5 \\ 4 & 0 \end{bmatrix} = 20$ $A_{22} = \det \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = 0$, $A_{23} = -\det \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = -4$

$A_{31} = \det \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix} = -5$ $A_{32} = -\det \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} = 10$, $A_{33} = \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 1$.

$\mathrm{Adj}(A) = \begin{bmatrix} 0 & 20 & -5 \\ 0 & 0 & 10 \\ 8 & -4 & 1 \end{bmatrix}$.

$\det(A) = 40$.

3 All these matrices will have eigenvectors $\vec{v}_1, \dots, \vec{v}_n$:

(a) $A^5\vec{v}_i = \underbrace{A A A A A \vec{v}_i}_{\underbrace{\lambda v_i}_{\lambda_i^2 v_i}} = \dots = \lambda_i^5 \vec{v}_i \Rightarrow$ eigenvalues $\lambda_1^5, \dots, \lambda_n^5$. 5

(b) Since $A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow$ multiplying by A^{-1} yields 5

$\vec{v}_i = \lambda_i A^{-1}\vec{v}_i \Rightarrow A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$. $\Rightarrow$ eigenvalues $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$.

(c) $(A + 3I_n)\vec{v}_i = A\vec{v}_i + 3\vec{v}_i = (\lambda + 3)\vec{v}_i \Rightarrow$ eigenvalues $\lambda_1 + 3, \dots, \lambda_n + 3$

(d) Similar argument : $10\lambda_1, \dots, 10\lambda_n$

(e) Similar argument , $\lambda_1^2 + 3\lambda_1 - 5, \dots, \lambda_n^2 + 3\lambda_n - 5$.

4 (a) $A - \lambda I = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - 4\lambda - 45$. 5

Roots : $\lambda_1 = 9, \ \lambda_2 = -5$

(b) $A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$ $\det(A - \lambda I) = \lambda^2 - 9\lambda + 32$.

No real roots.

Complex roots : $\dfrac{9 \pm i\sqrt{47}}{2}$

(c) $\det(A - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 & 1 & \dots & 1 \\ 1 & 1-\lambda & 1 & \dots & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \dots & 1-\lambda \end{vmatrix} \begin{matrix} \\ -\mathrm{I} \\ \dots \\ -\mathrm{I} \end{matrix} = \det \begin{vmatrix} 1-\lambda & 1 & 1 & \dots & 1 \\ \lambda & -\lambda & 0 & \dots & 0 \\ \lambda & 0 & -\lambda & \dots & 0 \\ \lambda & 0 & 0 & \dots & -\lambda \end{vmatrix}$ 10

$= \lambda^{n-1} \det \begin{vmatrix} 1-\lambda & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ 1 & 0 & 0 & \dots & -1 \end{vmatrix} \begin{matrix} -\mathrm{I}-\mathrm{II}-\mathrm{III}-\dots \\ \\ \\ \\ \end{matrix} = \lambda^{n-1} \det \begin{vmatrix} n-\lambda & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ 1 & 0 & 0 & \dots & -1 \end{vmatrix} = (-\lambda)^{n-1}(n - \lambda)$.

Eigenvalues: $\lambda = 0$ (multiplicity $n-1$), $\lambda = n$. 10

5 True. $\det(A) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \Rightarrow$ discriminant $\Delta = [\operatorname{tr}(A)]^2 - 4\det(A)$ will be non-negative if $\det(A) < 0$.

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$\Rightarrow$ there will be real roots.