Homework 11 Solutions

(1) Yes, e.g. $A = \left[\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 0 & -1 \\ & & 1 & 0 \end{array}\right]$ has the same eigenvalues as each block (rotation): $\underline{\underline{\pm i}}$.

(2) (a) $\begin{cases} \lambda_1 + \lambda_2 = tr(A) = 7 \\ \lambda_1 \lambda_2 = \det(A) = 12 \end{cases} \Rightarrow \underline{\lambda_1 = 3}, \ \underline{\lambda_2 = 4}$

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(b) No: any matrix similar to $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ has the same eigenvalues (thus same trace and determinant)

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For example, $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$.

(3) $\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 3$

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$\underline{\lambda = 1}$: $E_1 = \ker \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 2 \end{bmatrix}$

If $a \neq 0$:

$\begin{cases} a x_1 = 0 \Rightarrow x_1 = 0 \\ b x_1 + c x_2 + 2x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ c x_2 + 2x_3 = 0 \end{cases}$

$\begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = -\frac{c}{2} t \end{cases}$ Solution space: $E_1 = \left\{ \begin{bmatrix} 0 \\ t \\ -(c/2)t \end{bmatrix} ; t \in \mathbb{R} \right\}$

$= \operatorname{span} \begin{bmatrix} 0 \\ 1 \\ -c/2 \end{bmatrix}$.

~~dim~~

If $\underline{a \neq 0}$ then $\dim E_1 = 1 \Rightarrow$ <u>no</u> basis of eigenvectors.

let $a = 0$ $\quad E_1 = \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & c & 2 \end{bmatrix}$ $\quad b x_1 + c x_2 + 2x_3 = 0$

E_1 = plane $\Rightarrow$ $\underline{\dim E_1 = 2}$

$\underline{\lambda = 3}$: $\quad E_3 = \ker \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ b & c & 0 \end{bmatrix}$ $\quad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ \end{cases}$ $\quad E_3 = \operatorname{span} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\dim E_3 = 1}$.

$\dim E_1 + \dim E_2 = 2 + 1 = 3$

$\Rightarrow$ there is a basis of eigenvectors.

<u>Answer</u>: There is a basis of eigenvectors if and only if $a = 0$

(4) They must be the same. If they are not, then 10

$\sum$ geometric mult. $<$ $\sum$ algebraic mult $= n$

(see Theorem 7.3.7)

$\Rightarrow$ no basis of eigenvectors (Thm 7.3.4.b).

(5) (a) $\det(A - \lambda I) = \lambda^2 - 6\lambda + 9 = (3-\lambda)^2$. $\lambda_1 = \lambda_2 = 3$ 5

$A - 3\lambda = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow E_3 = \ker \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\dim E_3 = 1 < 2 \Rightarrow$ not diagonalizable

(b) $\det(A - \lambda I) = (1-\lambda)(2+\lambda)^2$ $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -2$ 10

Basis for E_1: $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for E_{-2}: $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$A = S^{-1} D S$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. (Other answers are possible)

(c) $\det(A - \lambda I) = (1-\lambda)(2+\lambda)^2$. $\lambda_1 = 1$, $\lambda_2 = \lambda_2 = -2$.

Basis for E_1: $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = -2$. $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \dim E_{-2} = 1 < 2 \Rightarrow$ not diagonalizable

(6) 10

True: Let $A = S^{-1} D S$. Since A is invertible, 0 is not an eigenvalue.

Hence $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is invertible, $D^{-1} = \begin{bmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_n \end{bmatrix}$

Moreover, S is invertible by the Diagonalization Theorem. Hence

$$A^{-1} = (S^{-1} D S)^{-1} = S^{-1} D^{-1} S.$$

Thus A^{-1} can be diagonalized.