Homework 12 Solutions

① Diagonalization yields (steps skipped):

$$A = S^{-1}DS, \qquad S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

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For $f(x) = x^3 - 4x^2 + 2x - 9$,

$$f(A) = S^{-1}\begin{bmatrix} f(5) & 0 \\ 0 & f(3) \end{bmatrix} S = S^{-1}\begin{bmatrix} 26 & 0 \\ 0 & -12 \end{bmatrix} S = \begin{bmatrix} 64 & 38 \\ -76 & -50 \end{bmatrix}$$

② (a) $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$

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$\lambda_1 = 2 + i, \quad \vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$\lambda_2 = 2 - i, \quad \vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$$A = S^{-1}DS, \qquad S^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

③ $\|\bar{u}+\bar{v}\|^2 = (\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v}) = \bar{u}\cdot\bar{u} + \bar{v}\cdot\bar{u} + \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{v} = \|\bar{u}\|^2 + 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2.$

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Similarly,

$\|\bar{u}-\bar{v}\|^2 = \|\bar{u}\|^2 - 2\bar{u}\cdot\bar{v} + \|\bar{v}\|^2$

$\Rightarrow \quad \underline{\|\bar{u}+\bar{v}\|^2 - \|\bar{u}-\bar{v}\|^2 = 4\bar{u}\cdot\bar{v}.}$

Hence $4\bar{u}\cdot\bar{v} = 3^2 - 1^2 = 8. \quad \Rightarrow \bar{u}\cdot\bar{v} = 2$

④ (a) $\lambda_1 = 8, \quad \bar{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$

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$\lambda_2 = 2, \quad \bar{v}_2 = \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}.$$

The answer $A = 8\bar{v}_1\bar{v}_1^T + 3\bar{v}_2\bar{v}_2^T$ is OK, too.

(5)

(a) True

(b) Not always – this is only true if $\|\vec{v}\| = 1$.

(c) True:

$$A^T = (S^{-1}DS)^T = S^T D^T (S^{-1})^T = S^{-1} D S \quad \left(\text{since } S \text{ is orthogonal, } D \text{ is diagonal}\right)$$

$$= A$$

(e) True (rank is not affected by multiplication by an invertible matrix, so rank $(S^{-1}AS) = \text{rank}(A)$)

(f) False: their eigenvalues may be different, so they won't be similar.

(5)

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$tr(A) = tr(B) = 2$

$\det(A) = \det(B) = 0$

But eigenvalues of A $(2,0,0)$ $\neq$ eigenvalues of B $(1,1,0)$

$\Rightarrow$ A, B are not similar.

(g) False: the Jordan block $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is invertible but not diagonalizable.