Homework 13 Solutions

1 (a) $q(\bar{x}) = \bar{x}^T A \bar{x}$, where $A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$. 10

Eigenvalues, eigenvectors: $\lambda_1 = 3$, $\bar{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\lambda_2 = 7$, $\bar{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\bar{y} = U\bar{x}$, $U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\Rightarrow$ $q(\bar{x}) = 3y_1^2 + 7y_2^2$

Eigenvalues of $A > 0 \Rightarrow$ positive definite

2 (b) $A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ singular values $\sigma_1 = \sqrt{4} = 2$, $\sigma_2 = \sqrt{1} = 1$ 5

Right Singular Vectors = Eigenvectors of $A^T A$: $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Left sing. vectors: $\bar{u}_1 = \frac{1}{\sigma_1} A\bar{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\bar{u}_2 = \frac{1}{\sigma_2} A\bar{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Thus $A = U\Sigma V^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. Eigenvalues: 18, 0 10

$\Rightarrow$ singular values $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, $\sigma_2 = 0$

Right singular vectors = eigenvectors of $A^T A$: $\bar{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Left singular vectors: $\bar{u}_1 = \frac{1}{\sigma_1} A\bar{v}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

$\bar{u}_2, \bar{u}_3$; eigenvectors for eigenvalue 0; can be found by choosing an orthonormal basis for $\bar{u}_1^\perp$, e.g.

$\bar{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$, $\bar{u}_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$

Thus
$$A = U\Sigma V^T = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(3) Let $A = U \Sigma V^T$ be SVD of A. Then 15

$$\det(A) = \det(U)\det(\Sigma)\det(V^T)$$

$$= \pm 1 \cdot \det(\Sigma) \qquad \text{(since } U, V^T \text{ are orthogonal)}$$

$$= \pm \sigma_1 \sigma_2 \dots \sigma_n \qquad \left(\text{since } \Sigma = \begin{vmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{vmatrix}\right) \quad \square.$$

(5) (a) T: follows from SVD: $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$

(b) False in general: if $n \neq m$ then A can't be orthogonal, but can have singular values $= 1$, e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

However, for square matrices ($n = m$) this is True.

$$A = U \underbrace{\Sigma}_{I_n} V = U V \text{ is orthogonal.}$$

(c) T: $A^T A = I \Rightarrow$ all eigenvalues of $A^T A$ are 1

(d) T: both equal # of non-zero singular values of A 5

$(A^T = \sum_{i=1}^{r} \sigma_i v_i u_i^T)$

(e) F: eigenvalues can be negative; singular values can't.

(Example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$). 5