Homework 13

Math 419, Winter 2013

1. Diagonalize the following quadratic forms. That is, find a change of variable \( \vec{y} = U \vec{x} \) so that the quadratic form becomes canonical (without cross-product terms). Determine whether each quadratic form is positive definite, positive semidefinite, or neither.
   (a) \( 5x_1^2 - 4x_1x_2 + 5x_2^2 \)
   (b) \( 8x_1^2 + 6x_1x_2 \)

2. For each matrix, find singular values and singular vectors (right and left). Find a singular value decomposition. Show all steps.
   (a) \[
   \begin{pmatrix}
   -3 & 0 \\
   0 & 0
   \end{pmatrix}
   \]
   (b) \[
   \begin{pmatrix}
   -2 & 0 \\
   0 & -1
   \end{pmatrix}
   \]
   (c) \[
   \begin{pmatrix}
   2 & 3 \\
   0 & 2
   \end{pmatrix}
   \]
   (b) \[
   \begin{pmatrix}
   1 & -1 \\
   -2 & 2 \\
   2 & -2
   \end{pmatrix}
   \]

3. Let \( A \) be a square matrix. Show that \( |\det A| \) is the product of the singular values of \( A \).

4. Let \( A \) be the rotation in the plane by angle \( \pi/4 \) counter-clockwise. Find the singular values and singular vectors (left and right) of \( A \).

5. Mark each statement True or False. Justify.
   (a) Any matrix of rank \( r \) can be expressed as a sum of \( r \) matrices of rank 1.
   (b) If all singular values of \( A \) equal 1 then \( A \) is orthogonal.
   (c) If \( A \) is orthogonal then all singular values of \( A \) equal 1.
   (d) The ranks of \( A^\top A \) and \( AA^\top \) are equal.
   (e) The eigenvalues of a symmetric matrix \( A \) are the same as singular values of \( A \).