Homework 3 Answers

(1) Elementary operations lead to the following rref:

(a)
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\quad} \cdots \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right].$$

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Hence $A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

(b)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\quad} \cdots \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/4 & -1/4 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right]$$

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not identity $\Rightarrow$ A is not invertible.

(c)
$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\quad} \cdots \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d)
$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\quad} \cdots$$ similarly to (b), $\text{rref}(A) \neq I_n \Rightarrow$ A is not invertible

(2) • Suppose $AB = AC$. If A is invertible, then multiplying both sides by A^{-1} we obtain

$$AA^{-1}B = A^{-1}AC$$
$$\Rightarrow B = C.$$

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(4+6)

• This is not true in general if A is not invertible. There can be many examples, one of which is:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = AC = 0$ but $B \neq C$.

(3) $A = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & v_1 \\ 0 & 1 & \cdots & 0 & v_2 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & v_{n-1} \\ \hline u_1 & u_2 & \cdots & u_{n-1} & 1 \end{array}\right]$.

We bring A to rref by subtracting from the last (n'th) row $u_1 \cdot (1^{st}$ row$)$ then $u_2 \cdot (2^{nd}$ row$), \dots, u_{n-1} \cdot ((n-1)$'th row$)$.

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This results in

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & v_1 \\ 0 & 1 & \cdots & 0 & v_2 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 1 & v_{n-1} \\ \hline 0 & 0 & \cdots & 0 & 1-u_1v_1-u_2v_2-\dots-u_nv_n \end{array}\right]$$

If $\lambda := 1-u_1v_1-u_2v_2-\dots-u_nv_n \underline{=0}$ then $\text{rref}(A) \neq I_n$ (due to the zero row), and thus A is <u>not</u> invertible.

If $\lambda \neq 0$ then we can multiply n'th row by $\frac{1}{\lambda}$, and then we continue: subtract $v_1 \cdot (n$'th row$)$ from 1^{st} row,

$v_2 \cdot (n$'th row$)$ from 2^{nd} row

$v_{n-1} \cdot (n$'th row$)$ from $(n-1)^{th}$ row

$\Rightarrow \text{rref}(A) = I_n \Rightarrow A$ is invertible.

Finally, $\lambda = 1 - \vec{u} \cdot \vec{v}$.

<u>Answer</u>: A is invertible if and only if $\boxed{\vec{u} \cdot \vec{v} \neq 1}$.

(4) (a) • $\text{Im}(A) = \text{span}(\vec{v}_1, \vec{v}_2)$ where $\vec{v}_1 = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$

10 = (a)

Note that $\vec{v}_2 = -\frac{4}{3}\vec{v}_1 \Rightarrow \text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{v}_1) = \left\{ \begin{bmatrix} 3t \\ -9t \end{bmatrix} : t \in \mathbb{R} \right\}$

line in $\mathbb{R}^2$.

• $\ker(A) = ?$ $\quad A\vec{x} = 0 \quad \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 9 & 12 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & 0 \\ 9 & 12 & 0 \end{array}\right] -9 \cdot I$

$\to \left[\begin{array}{cc|c} 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - \frac{4}{3}x_2 = 0 \quad t := x_2 \Rightarrow x_1 = \frac{4}{3}t$

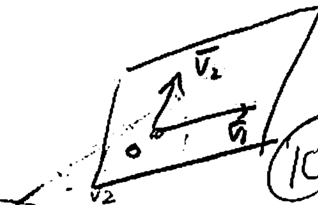
$\ker A = \left\{ \begin{bmatrix} \frac{4}{3}t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$ ← line in $\mathbb{R}^2$.

(b) One quickly checks that A is invertible (since $\mathrm{rref}(A) = I_n$) 5 = (b)

$\Rightarrow \mathrm{Im}(A) = \mathbb{R}^2, \quad \ker(A) = \{\vec{0}\}.$

(c) Similarly as in (B), A is invertible $\Rightarrow$

$\mathrm{Im}(A) = \mathbb{R}^3, \quad \ker(A) = \{\vec{0}\}$



(d) $\mathrm{Im}(A) = \mathrm{span}(\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$; $\vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$ 10 (d)

$\vec{v}_1, \vec{v}_2$ are not colinear $\Rightarrow$

$$\mathrm{Im}(A) = \left\{ c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \Leftarrow \text{Plane in } \mathbb{R}^3.$$

$\mathrm{Ker}(A) = ?$ $\qquad A\vec{x} = 0.$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ -1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \begin{array}{l} \\ +\mathrm{I} \\ -2\mathrm{I} \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & -12 & 0 \end{array}\right] \xrightarrow[\times \frac{1}{12}]{+4\cdot\mathrm{III}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \ker(A) = \{\vec{0}\}$$

(5) For example, $A = RP$ where P is the orthogonal projection onto x-axis, R = rotation by 90° counter-clockwise.

$\mathrm{Im}\, P = \{x\text{-axis}\}, \Rightarrow \mathrm{Im}(RP) = \{\text{rotation of } x\text{-axis}\} = \{y\text{-axis}\}.$

$\ker P = \{y\text{-axis}\}$, R is one-to-one $\Rightarrow \ker(RP) = (y\text{-axis})$

In matrix form, 10

$$A = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{P} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$