HOMEWORK 4 Answers

1 (a) $\text{im}(A) = \text{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$. For example, the following matrix works; 5

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

since all columns are $\begin{bmatrix}1\\-1\\1\end{bmatrix} \Rightarrow \text{im}(A) = \text{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$

(b) For example, an orthogonal projection onto z-axis works.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\ker(A) = (x\text{-axis}) \Rightarrow A\begin{bmatrix}1\\0\end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ 5

$\Rightarrow a = c = 0.$

$\text{im}(A) = \text{span}\left(\begin{bmatrix}a\\c\end{bmatrix}, \begin{bmatrix}b\\d\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}b\\d\end{bmatrix}\right)$

Given: $\text{im}(A) = (\text{line } y = x) = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$.

Hence we can choose $b = d = 1$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2 $\mathbb{R}^m \xrightarrow[A]{} \mathbb{R}^p \xrightarrow[B]{} \mathbb{R}^n$ 10

(a) If $\vec{x} \in \ker(A)$ then $A\vec{x} = \vec{0} \Rightarrow BA\vec{x} = B\vec{0} = \vec{0} \Rightarrow \vec{x} \in \ker(BA)$.

Thus $\boxed{\ker(BA) \supset \ker(A)}$

These sets are not necessarily equal, e.g. if $B = 0$, A invertible then $BA = 0 \Rightarrow \ker(BA) = \mathbb{R}^m$,
$\ker(A) = \{\vec{0}\}$

(b) No relation in general. For example, if $m = 1$, $p = n = 2$,
$A = \begin{bmatrix}1\\0\end{bmatrix}$, $B = \text{rotation by } 90^\circ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \text{im}(A) = (x\text{-axis})$,
$\text{im}(BA) = (y\text{-axis})$. (due to rotation).

(3) (a) $A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & 7 & 5 \end{bmatrix}$ $\quad rref(A) = I_3$ (by usual Gaussian elimination) (5)

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ form a basis of R^3

($\Leftrightarrow$ span R^3, lin. indep.)

(b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad rref(A) \neq I_3$

(one can also see that $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$ – a linear relation).

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not linearly indep

($\Rightarrow$ do not span R^3, do not form a basis)

(c) Similar answer (since $\vec{v}_2 = \vec{0}$)

(d) $rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$ $\Rightarrow$ similar answer

(e) $\vec{v}_2$ is not colinear with $\vec{v}_1$ $\Rightarrow$ $\vec{v}_1, \vec{v}_2$ are linearly independent. (5)

They don't span $\mathbb{R}^3$ (one needs at least 3 vectors for that)

$\Rightarrow$ they don't form a basis

(f) These span R^3 (since the first three already span $\mathbb{R}^3$ by (a)) (5)

They are not lin. indep (one must have ≤ 3 vectors for that)

$\Rightarrow$ they don't form a basis.

(4) (a) Choose two linearly independent vectors from the plane, e.g (10)

$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$

Since dim(plane) = 2, $\vec{v}_1, \vec{v}_2$ form a basis of the plane.

(b) $\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\quad \begin{cases} 2c_1 = 6 \\ 2c_2 = 2 \\ -c_1 + 3c_2 = 0 \end{cases} \Rightarrow c_1 = 3, \; c_2 = 1.$

$$\boxed{\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}}$$

5 (a) <u>No</u> : $\vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix} \notin W$ since all vectors $\vec{v}$ in W have the third coord $= 1 \neq 0$

(b) <u>Yes</u> : $W = \operatorname{im}(A)$, where $A = \begin{bmatrix}1 & 0\\0 & 1\\1 & -1\\1 & 2\end{bmatrix}$ 10

indeed, $A\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a\\b\\a-b\\a+2b\end{bmatrix}$

(c) <u>No</u> : for instance, $\vec{v} = (1, 0, 0) \in W$

$\vec{w} = (0, 1, 0) \in W$

But $\vec{v} + \vec{w} = (1, 1, 0) \notin W$

(third coord is not $1 \cdot 1$).