Homework 5 Solutions

1 a) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank$(A) = 3 \Rightarrow$ by RNT (rank-nullity theorem), dim$(\ker A) = 7-3 = 4$ 5

b) Does not exist: $\ker(A) \subset \mathbb{R}^3 \Rightarrow \dim(\ker A) \le 3$.

c) Does not exist. Contradicts RNT $(4+3 \neq 6)$ 5

d) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank$(A) = 1 \Rightarrow$ by RNT, dim$(\ker A) = 6-1 = 5$.

e) $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C}$ 5

2 (a) No: if $f, g \in W$ then $f(0) = 1$, $g(0) = 1 \Rightarrow (f+g)(0) = f(0) + g(0) = 2 \neq 1$ 5

$\Rightarrow f+g \notin W$

(b) Yes: if $f, g \in W$ then $f(0) = g(0) = 0 \Rightarrow (f+g)(0) = f(0) + g(0) = 0$. $\Rightarrow f+g \in W$

$(kf)(0) = k \cdot f(0) = k\,0 = 0. \Rightarrow kf \in W$

(c) Yes: • If $f, g \in W$ then $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 0$ 5

$\Rightarrow \int_0^1 (f+g)(x)\,dx = 0 + 0 = 0. \Rightarrow f+g \in W$.

• $\int_0^1 kf(x)\,dx = k \cdot 0 = 0 \Rightarrow kf \in W$.

(d) No: let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $k = 0 \Rightarrow kA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible

$(\Rightarrow A \in W)$ $\Rightarrow kA \notin W$

3. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ $BA = 0$: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = 0$

(a)

$$\Leftrightarrow \begin{cases} A_{11} + 2A_{21} = 0 \\ A_{12} + 2A_{22} = 0 \\ 2A_{11} + 4A_{21} = 0 \\ 2A_{12} + 4A_{22} = 0 \end{cases} \text{ redundant} \quad \Leftrightarrow \quad \begin{cases} A_{11} = -2A_{21} \\ A_{12} = -2A_{22} \end{cases} \text{ free var's. } (*)$$

Choosing e.g. $(A_{21}=1,\ A_{22}=0)$ and $(A_{21}=0,\ A_{22}=1)$

we obtain the two matrices

$$\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$$

which are not colinear $\Rightarrow$ linearly independent.

From $(*)$ we see that $\boxed{\dim(V) = 2}$ $\Rightarrow$ these matrices form a basis of V.

(b) $f(x) = a + bx + cx^2$ $\quad f(0) = 0 \Leftrightarrow a + b\cdot 0 + c\cdot 0^2 = 0 \Leftrightarrow \underline{a = 0}$.

$\Rightarrow$ $V = \{bx + cx^2.\ b, c \in \mathbb{R}\}$

$\underline{x, x^2}$ form a basis for V; $\quad \underline{\dim(V) = 2}$.

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(c) Similarly, f even $\Leftrightarrow$ $a + bx + cx^2 = a - bx + cx^2$ for all x.

$\Leftrightarrow b = 0$.

$V = \{a + cx^2.\ a, c \in \mathbb{R}\}$.

$\underline{1, x^2}$ form a basis for V, $\quad \underline{\dim V = 2}$.

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4 (a) $\underline{\mathrm{Im}(T) = \mathbb{R}}$, as any number $c \in \mathbb{R}$ can be represented as

$$c = \int_1^4 f(x)\,dx \quad \text{for some } f \in P_2[x]$$

(e.g. choose $f(x) = c$)

$\underline{\mathrm{rank}(T)} = \dim(\mathbb{R}) = 1$.

$\ker(T) = ?$ $\quad \int_1^4 f(x)\,dx = 0. \qquad f = a + bx + cx^2$

$$\Rightarrow \left[a + \frac{bx^2}{2} + \frac{cx^3}{3}\right]\Big|_1^4 = 0$$

$$3 + \frac{15}{2}b + \frac{63}{3}c = 0$$

Hence $\underline{\ker(T) = \{a + bx + cx^2 : \ 3 + \frac{15}{2}b + \frac{63}{3}c = 0\}}$.

(c) $f(x) = a + bx + cx^2 \qquad T(f) = x(b + 2cx) = bx + 2cx^2$ 10

$\underline{\mathrm{Im}(T)} = \{bx + 2cx^2 . \ b, c \in \mathbb{R}\} = \underline{\mathrm{span}(x, x^2)}$

$\Rightarrow \mathrm{rank}(T) = 2$

$\ker(T) = ? \qquad bx + 2cx^2 = 0$ for all x

$\Leftrightarrow b = c = 0$.

$\underline{\ker(T)} = \{a : a \in \mathbb{R}\} = \underline{\mathrm{span}(1)}$

5 For instance, $\{x^2, 1+x^2, x+x^2\}$. 10

One can check that these vectors are linearly independent (and thus form a basis of $P_2[x]$)

either directly, or by looking at the coefficients w.r to $\mathcal{B} = \{1, x, x^2\}$:

$$[x^2]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad [1+x^2] = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad [x+x^2]_{\mathcal{B}} = \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad (*)$$

The matrix $\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}$ is invertible (check!)

$\Rightarrow$ the coefficient vectors in $(*)$ are linearly independent

$\Rightarrow$ the elements x^2, $1+x^2$, $x+x^2$ are lin independent.