Homework 5
Math 419, Winter 2013

1. In each part, find examples of matrices as stated, or explain why no such examples exist:
   (a) $3 \times 7$ matrix with $\dim(\ker(A)) = 4$.
   (b) $8 \times 3$ matrix with $\dim(\ker(A)) = 4$.
   (c) $4 \times 6$ matrix with $\dim(\ker(A)) = 4$ and $\text{rank}(A) = 3$.
   (d) $4 \times 6$ matrix with $\dim(\ker(A)) = 5$ and $\text{rank}(A) = 1$.
   (e) An invertible $3 \times 3$ matrix $C$ which can be expressed as $C = BA$ for some $5 \times 3$ matrix $A$ and $3 \times 5$ matrix $B$.

2. Which of the following sets are linear spaces? If you believe that it is a linear space, do not check all the axioms – just check that the set is closed under addition and multiplication by scalars. If you believe that the set is not a linear space, explain why (find an example that violates one of the requirements).
   (a) The set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$.
   (b) The set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.
   (c) The set of all continuous functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(x) \, dx = 0$.
   (d) The set of all invertible $2 \times 2$ matrices.

3. In each part, construct a basis for the linear space $V$, and compute the dimension of $V$.
   (a) $V$ is the set of all $2 \times 2$ matrices $A$ satisfying $BA = 0$, where $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
   (b) $V$ is the set of quadratic polynomials $f$ such that $f(0) = 0$.
   (c) $V$ is the set of even quadratic polynomials $f$, i.e. such that $f(x) = f(-x)$ for all $x$.

4. Find the image, kernel and rank of the following linear transformations:
   (a) $T(f) = \int_1^4 f(x) \, dx$ from $P_2[x]$ to $\mathbb{R}$.
   (b) $T(f) = f(8)$ from $P_2[x]$ to $\mathbb{R}$.
   (c) $T(f) = xf'(x)$ from $P_2[x]$ to $P_2[x]$. (For example, if $f(x) = x - x^2$ then $(T(f))(x) = x(1-2x) = x - 2x^2$.)

5. One basis for the linear space $P_2[x]$ is formed by the monomials $1$, $x$, $x^2$. Construct a different basis for $P_2[x]$ whose elements are all quadratic functions (i.e. they are all of the form $a + bx + cx^2$ with $c \neq 0$.) Explain why your set is a basis.