Homework 6 Answers. 5

1 (a) In the Basis $\mathcal{B} = \{1, x, x^2\}$:

$T(1) = 1$, $T(x) = x - 2$, $T(x^2) = x^2 - 4x$

$\Rightarrow$ the matrix : $T_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

Since the columns are linearly independent, $T_{\mathcal{B}}$ is invertible

$\Rightarrow$ T is an isomorphism.

b) No since $\dim(\mathbb{R}^3) = 3$, $\dim(\text{plane}) = 2$

$\Rightarrow$ $\mathbb{R}^3$ is not isomorphic to the plane.

c) Matrix : 5

$$T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Columns are linearly dependent (use rref or observe that sum of the columns $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$).

$\Rightarrow$ T is not an isomorphism.

(d) Yes : T is the left shift by 1.

$T^{-1} : C[0,1] \to C[1,2]$: $T(f) = f(t-1)$

is the right shift by 1.

2 a) Choose bases for V and W, e.g.

$$\mathcal{B} = \Big\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{=\vec{v}_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{=\vec{v}_2} \Big\} \text{ of } V,$$

$$\mathcal{U} = \Big\{ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=\vec{w}_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{=\vec{w}_2} \Big\} \text{ of } W.$$

Define $T: V \to W$ as

$$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\vec{w}_1 + c_2\vec{w}_2.$$

T is well defined since every $\vec{v} \in V$ has a unique repres. as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$.
T is invertible since

$$T^{-1}(c_1\vec{w}_1 + c_2\vec{w}_2) = c_1\vec{v}_1 + c_2\vec{v}_2$$

b). Yes $\quad T(a + cx^2) = \begin{bmatrix} a \\ c \end{bmatrix} \quad T: P_2[x] \to \mathbb{R}^2$ 5

T is a coordinate transf. $\Rightarrow$ isomorphism.

c) No : by rank-nullity theorem, 5

$$\dim(V) + \dim(W) = 3$$

Hence $\dim(V) \neq \dim(W)$ (since both are integers)

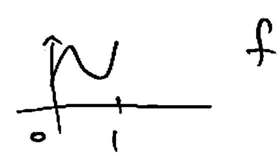
$\Rightarrow$ V and W are not isomorphic

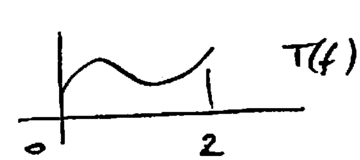
d) Yes, $T: V \to W$

$$T(f) = f(t/2)$$

is an isomorphism.

$$T^{-1}(f) = f(2t)$$





3 a). $\mathcal{U} = \{\vec{e}_1, \vec{e}_2\}$, $\mathcal{B} = \{ \begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}-3\\5\end{bmatrix} \}$.

10 (total for a+b)

$L_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix}-1 & -3\\ 2 & 5\end{bmatrix}$, $L_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix}-1 & -3\\ 2 & 5\end{bmatrix}^{-1} = \begin{bmatrix}5 & 3\\ -2 & -1\end{bmatrix}$

b) $\begin{bmatrix}2\\-5\end{bmatrix}_{\mathcal{B}} = L_{\mathcal{U}\to\mathcal{B}} \begin{bmatrix}2\\-5\end{bmatrix}_{\mathcal{U}} = \begin{bmatrix}5 & 3\\ -2 & -1\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}-5\\1\end{bmatrix}$

4 a) W. r. to standard basis $\mathcal{B} = \{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \}$:

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$$T_{\mathcal{B}} = \begin{bmatrix}0 & 2\\ 3 & -1\end{bmatrix}$$

W. r. to $\mathcal{U} = \{ \underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix}2\\5\end{bmatrix}}_{\vec{u}_2} \}$

$T_{\mathcal{U}} = S T_{\mathcal{B}} S^{-1}$ where $S = S_{\mathcal{B}\to\mathcal{U}}$ (see Thm. 4.3.5)

$S_{\mathcal{U}\to\mathcal{B}} = \begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$, $S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}^{-1} = \begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}$

$T_{\mathcal{U}} = \begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}0 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix} = \begin{bmatrix}-30 & -48\\ 18 & 29\end{bmatrix}$

5. $\vec{v}_1 = \begin{bmatrix}2\\5\\2\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}3\\-2\\2\end{bmatrix}$. $\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow$ orthogonal

$\vec{v}_1 = \sqrt{2^2+5^2+2^2} = \sqrt{33}$, $\vec{v}_2 = \sqrt{3^2+2^2+2^2} = \sqrt{17}$. Not orthonormal.

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$

$P\vec{x} = (\vec{u}_1\cdot\vec{x})\vec{u}_1 + (\vec{u}_2\cdot\vec{x})\vec{u}_2 = \frac{1}{33}(\vec{v}_1\cdot\vec{x})\vec{v}_1 + \frac{1}{17}(\vec{v}_2\cdot\vec{x})\vec{v}_2$

$= \frac{1}{33}\left(\begin{bmatrix}2\\5\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\3\end{bmatrix}\right)\begin{bmatrix}2\\5\\2\end{bmatrix} + \frac{1}{17}\left(\begin{bmatrix}3\\-2\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\3\end{bmatrix}\right)\begin{bmatrix}3\\-2\\2\end{bmatrix} = \cdots$

6 $\left.\begin{array}{l}\vec{u} = \begin{bmatrix}x\\y\\z\end{bmatrix}\\ \vec{v} = \begin{bmatrix}1\\-2\\4\end{bmatrix}\end{array}\right\} \Rightarrow x - 2y + 4z = \vec{u}\cdot\vec{v}$. Cauchy-Schwartz:

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$\vec{u}\cdot\vec{v}$ is maximal when $\vec{u}$ is colinear with $\vec{v}$,

ie. $\begin{bmatrix}x\\y\\z\end{bmatrix} = k\begin{bmatrix}1\\-2\\4\end{bmatrix}$ But $\|\vec{u}\| = 1 \Rightarrow$

$|k|\left\|\begin{bmatrix}1\\-2\\4\end{bmatrix}\right\| = 1 \Rightarrow k = \pm\frac{1}{\left\|\begin{bmatrix}1\\-2\\4\end{bmatrix}\right\|} = \pm\frac{1}{\sqrt{21}}$

Answer: $\vec{u} = \pm\begin{bmatrix}1/\sqrt{21}\\ -2/\sqrt{21}\\ 4/\sqrt{21}\end{bmatrix}$