Homework 7 Solutions

(1) a $\mathcal{B} = \{1, x, x^2\}$. $T(f) = f'' + 4f'$. (10)

$$T_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [T(1)]_{\mathcal{B}} & [T(x)]_{\mathcal{B}} & [T(x^2)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}.$$

$T(1) = 0$, $T(x) = 4$, $T(x^2) = 2 + 8x$.

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^2)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}.$$

$$T_{\mathcal{B}} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $\vec{x} := \vec{v}_4$ must satisfy $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = \vec{x} \cdot \vec{v}_4 = 0$ (10)

This amounts to solving

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases} \quad \text{(omitting coefficients } 1/2\text{)}$$

The solutions are $\vec{x} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

Since $\vec{x}$ must be a unit vector, $\|\vec{x}\| = 2|t| = 1 \Rightarrow |t| = 1/2$

Hence there are <u>two</u> ~~possible~~ $\vec{v}_4$:

$$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \text{ and } \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

(3) (a) False : for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has $Im(A) = (x\text{-axis})$
$Ker(A) = (\text{line } y = -x)$.

(b) False

(c) False : $V = (x\text{-axis})$ and $W = (y\text{-axis})$ in $\mathbb{R}^3$ are orthogonal subspaces, but $V^\perp = (yz\text{-plane}) \neq W$.

(d) False. $V = (x\text{-axis})$, $W = (y\text{-axis})$ in $\mathbb{R}^3$.
$V^\perp = (yz\text{-plane})$, $W^\perp = (xy\text{-plane})$
$V^\perp, W^\perp$ are not orthogonal since they share a line (y-axis)

(4) $W^\perp$ consists of the vectors $\vec{x} \in \mathbb{R}^4$ such that $\begin{cases} \vec{x} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 0 \\ \vec{x} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 0. \end{cases}$ (10)

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 = 0 \end{cases}$$

Solving yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s + 2t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $W^\perp$.

(5) By Pythagoras' Theorem (for more than 2 vectors), since $3\vec{v}_1, -2\vec{v}_2, 7\vec{v}_3, -\vec{v}_4$ and $\vec{v}_5$ are orthogonal $\Rightarrow$ (10)

$$\|\vec{v}\|^2 = \|3\vec{v}_1\|^2 + \|-2\vec{v}_2\|^2 + \|7\vec{v}_3\|^2 + \|-\vec{v}_4\|^2 + \|\vec{v}_5\|^2$$
$$= 9\underbrace{\|\vec{v}_1\|^2}_{=1} + 4\underbrace{\|\vec{v}_2\|^2}_{=1} + 49\underbrace{\|\vec{v}_3\|^2}_{=1} + \underbrace{\|\vec{v}_4\|^2}_{=1} + \underbrace{\|\vec{v}_5\|^2}_{=1} = 64.$$

Hence $\|\vec{v}\| = 8$.