Homework 9 Solutions

1 (a) True: for each $\vec{v} \in \mathbb{R}^n$,

$$\| UV\vec{v} \| = \| U(V\vec{v}) \| = \| V\vec{v} \| \quad (U \text{ orthogonal})$$
$$= \| \vec{v} \| \quad (V \text{ orthogonal})$$

Thus UV is orthogonal

(b) False: the zero matrix is not orthogonal

(c) False: $\det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1$.

(d) True: the columns $\vec{u}_i$ form an orthonormal basis.

$\Rightarrow \| \vec{u}_i \| = 1$. But $\| \vec{u}_i \|^2 = u_{i1}^2 + \dots + u_{in}^2$

Hence $u_{ij}^2 \leq 1$ for all j

$\Rightarrow -1 \leq u_{ij} \leq 1$.

(e) True. $A^T = A^{-1}$, and A^{-1} is orthogonal by Prop. 5.3.4 5

2 $A^{-1}\vec{e}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$. Hence this is the 3^{rd} column of A^{-1}. 10

We can find two vectors that together with $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ form an orthonormal basis of $\mathbb{R}^3$, e.g. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}.$$

Since $A^{-1} = A^T \Rightarrow A = (A^{-1})^T = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

3 (a) $A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$, $A^T\vec{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ 10

$$\hat{x} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{24}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) $A^TA = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$, $A^T\vec{b} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$.

$$\hat{x} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{11}\begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix}\begin{bmatrix} -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

4 (a) Expand in 3rd row.

$$\det(A) = -(-2)\det\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = -2$$

(b) Expand in 3rd row, then along 1st row. 5

$$\det(A) = (-1)^{3+1}\cdot 2\det\begin{bmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{bmatrix} = 2(-1)^{1+3}\,5\det\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} = 10$$

(c) Expand in 1st column, then in 1st column again ⇒

$$\det(A) = 3\cdot 2\cdot\det\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = 3\cdot 2\,(-12) \quad \text{(by part (a))}$$ 5

$$= -12$$