Lec 21
04/04

7.5 Complex eigenvalues.

Eigs = roots of char. poly
$\det(A - \lambda I) = 0$ ~~char equation~~

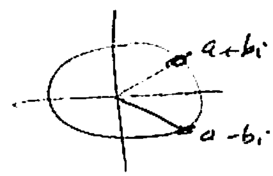
Thm (The fundamental thm of algebra).

- Any poly $p(x)$ of degree n {with complex or real coeff's}
  has exactly n ~~solutions~~ roots (counting multiplicity), real or complex.

  Thus
  $$p(x) = a_0 (x - x_1)(x - x_2) \cdots (x - x_n)$$

  Some roots x_i can be the same.

- Moreover, if $p(x)$ has real coeff's then for each non-real root $x_i = a + bi$ $(b \neq 0)$ there exists a conjugate root $x_j = \overline{x_i} = a - bi$



$\Downarrow$

Cor • Any $n \times n$ matrix A with complex or real entries has exactly n eigenvalues $\lambda_1, \dots, \lambda_n$ (counting alg. multiplicities)

• Moreover, ~~if $\exists$ eigen~~ for each non-real eig $\lambda = a + bi$
$\exists$ conjugate eig. $\overline{\lambda} = a - bi$.

Hence: always true that

$$\boxed{\begin{aligned} \det(A) &= \lambda_1 \cdots \lambda_n \\ \operatorname{tr}(A) &= \lambda_1 + \cdots + \lambda_n \end{aligned}}$$

Ex (a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$\lambda_1 = -i$, $\lambda_2 = i$ (Compare w/ F.T.A)

-i) $A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$

$(A - iI)x = 0$ $\quad \begin{cases} x_1 - x_2 = 0 \\ x_1 + i x_2 = 0 \end{cases}$ $\quad x_1 = 1, \ x_2 = i$

$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

i) $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ $\quad \begin{cases} i x_1 + x_2 = 0 \\ x_1 - i x_2 = 0 \end{cases}$ $\quad x_1 = i$, $x_2 = 1$ $\quad x = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Thus: $\lambda_1 = -i, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda_2 = i, \quad \vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Diagonalization: $A = SDS^{-1}$, $\quad D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$, $\quad S = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (Jordan block) – still not diagonalizable.

• In view of the fundamental theorem of algebra, we see

<u>Thm</u>: • Any $n \times n$ matrix has exactly n eigenvalues (counting multiplicity) $\lambda_1, \lambda_2, \cdots, \lambda_n$.

• If A is a real matrix with a complex eigenvalue $\lambda = a+bi$ and corresponding complex eigenvector $\vec{v}+i\vec{w}$, then $\bar{\lambda} = a-bi$ is also an eigenvalue with eigenvector $\vec{v}-i\vec{w}$.

• (As we mentioned earlier) $\lambda_1 \cdots \lambda_n = \det A$

$\lambda_1 + \cdots + \lambda_n = \operatorname{tr} A$

3. Canonical Forms

• After introducing complex eigenvalues/eigenvectors, one can diagonalize more matrices: the invertible matrix A may contain complex vectors, and the diagonal matrix D may contain complex eigenvalues.

e.g. $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1}$.

$\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{2}i}{3} & \frac{1+\sqrt{2}i}{3} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{2}i & 0 \\ 0 & 1-\sqrt{2}i \end{pmatrix}\begin{pmatrix} \frac{1-\sqrt{2}i}{3} & \frac{1+\sqrt{2}i}{3} \\ 1 & 1 \end{pmatrix}^{-1}$.

As before, we have

<u>Thm</u>: A is diagonalizable over $\mathbb{C}$ $\Leftrightarrow$ Exists a complex eigenbasis for A.

Canonical forms

<u>Note</u>: "Most" polynomials of degree n have n distinct roots

$\Rightarrow$ "Most" $n \times n$ matrices have n distinct eigenvalues

$\Rightarrow$ "Most" $n \times n$ matrices are diagonalizable.

<u>However</u>: Matrices like $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are not diagonalizable even over $\mathbb{C}$.

• <u>Question</u>: What is the simplest possible form of A if A is not diagonalizable?

<u>Observation</u>: If A is not diagonalizable, A must have at least one eigenvalue whose algebraic multiplicity is greater than 1.

e.g. for $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\lambda = 0$ has algebraic multiplicity 4.

<u>Fact</u>: Such a "1's over diagonal" matrix form fundamental blocks of the simplest matrix that is similar to a non-diagonalizable matrix.

Thm (Jordan canonical form). Any matrix is similar (over $\mathbb{C}$) to a matrix of the form

$$\begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & \\ & \ddots & 1 \\ & & \lambda_1\end{matrix}} & & & & 0 \\ & \boxed{\begin{matrix}\lambda_1 & 1 \\ & \lambda_1\end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix}\lambda_2 & 1 & \\ & \ddots & 1 \\ & & \lambda_2\end{matrix}} & \\ & & & & \ddots \\ 0 & & & & \boxed{\begin{matrix}\lambda_k & 1 & \\ & \ddots & 1 \\ & & \lambda_k\end{matrix}} \end{pmatrix}$$

where each block $\begin{pmatrix}\lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda\end{pmatrix}$ is called a Jordan block.

- In the case each Jordan block is 1×1, A is diagonalizable.
- The total number of Jordan blocks (including 1×1 blocks) $= \sum_\lambda$ (geometric multiplicities)

Note: The matrix ~~A~~ S that ~~conjugate~~ brings A into its Jordan form consists of eigenvectors as well as generalized eigenvectors, that is, solutions to the equations ~~$S^{-1}AS$~~ $A = SJS^{-1}$

$$(A-\lambda I)^k \vec{w} = \vec{v}$$

for some k and some eigenvector $\vec{v} \in E_\lambda$.

- **Question**. What is the simplest form of $S^{-1}AS$ if we can only use real matrix S?

For simplicity we only look at the case A is diagonalizable (over $\mathbb{C}$).

Theorem. Suppose A is a real 2×2 matrix with eigenvalues $a\pm bi$ and corresponding eigenvectors $\vec{v}\pm i\vec{w}$. Then with $S=\begin{bmatrix} | & | \\ \vec{v} & \vec{w}\end{bmatrix}$ we have $S^{-1}AS=\begin{bmatrix} a & b \\ -b & a\end{bmatrix}$.

Reason: $A(\vec{v}+i\vec{w}) = (a+ib)(\vec{v}+i\vec{w}) = (a\vec{v}-b\vec{w}) + i(b\vec{v}+a\vec{w})$

$\Rightarrow A\vec{v} = a\vec{v}-b\vec{w}, \quad A\vec{w} = b\vec{v}+a\vec{w}$

$\Rightarrow AS = A[\vec{v}\ \vec{w}] = [a\vec{v}-b\vec{w} \quad b\vec{v}+a\vec{w}]$

$S\begin{bmatrix} a & b \\ -b & a\end{bmatrix} = [\vec{v}\ \vec{w}]\begin{bmatrix} a & b \\ -b & a\end{bmatrix} = [a\vec{v}-b\vec{w} \quad b\vec{v}+a\vec{w}]$

$\Rightarrow S^{-1}AS = \begin{bmatrix} a & b \\ -b & a\end{bmatrix}$. $\square$

Example. $\begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ 1 & 0\end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ -3 & 2\end{pmatrix}\begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ 1 & 0\end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1\end{pmatrix}$.

In general. If A is diagonalizable with eigenvalues $\lambda_1, \bar{\lambda}_1, \underset{a_2+ib_2}{\lambda_2}, \bar{\lambda}_2, \dots, \underset{a_k+ib_k}{\lambda_k}, \bar{\lambda}_k, \lambda_{k+1}, \dots, \lambda_l$, then A is similar over $\mathbb{R}$ to a matrix

$$\begin{pmatrix} a_1 & b_1 & & & & & 0 \\ -b_1 & a_1 & & & & & \\ & & \ddots & & & & \\ & & & a_k & b_k & & \\ & & & -b_k & a_k & & \\ & & & & & \lambda_{k+1} & \\ 0 & & & & & & \ddots \ \lambda_l \end{pmatrix}$$

4.

2. Functions of Matrices

- As we have seen in lecture 19, <u>if A is diagonalizable</u> ($\Leftrightarrow$ "enough eigenvectors"), then to calculate a high power of A is simple if we are using the eigenbasis. Alternatively,

$$D = S^{-1}AS \Rightarrow A = S(S^{-1}AS)S^{-1} = SDS^{-1} = S\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}S^{-1}$$

$$\Rightarrow A^2 = (SD\underbrace{S^{-1})(S}DS^{-1}) = SD^2S^{-1} = S\begin{pmatrix}\lambda_1^2 & & 0\\ & \ddots & \\ 0 & & \lambda_n^2\end{pmatrix}S^{-1}$$

$$\vdots$$

$$A^N = \underbrace{(SD\overbrace{S^{-1})(S}^{\text{cancel}}DS^{-1})\cdots(SDS^{-1})}_{N \text{ times}} = SD^NS^{-1} = S\begin{pmatrix}\lambda_1^N & & 0\\ & \ddots & \\ 0 & & \lambda_n^N\end{pmatrix}S^{-1}.$$

- Now let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial, then we can define

$$p(A) = a_0I_n + a_1A + \cdots + a_nA^n.$$

<u>If A is diagonalizable</u>, i.e. $A = SDS^{-1}$, then

$$p(A) = S\cdot a_0I_n\cdot S^{-1} + S\cdot a_1D\,S^{-1} + \cdots + S\,a_nD^nS^{-1} = S(p(D))\cdot S^{-1} = S\begin{pmatrix}p(\lambda_1) & & 0\\ & \ddots & \\ 0 & & p(\lambda_n)\end{pmatrix}S^{-1}.$$

So $p(A)$ is also diagonalizable, with the same eigenbasis as A, and eigenvalues become $p(\lambda_1), \dots, p(\lambda_n)$.

- One can even define more complicated functions using power series, e.g.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \rightsquigarrow e^A := I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \rightsquigarrow \sin A := A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots$$

→ or with

~~$f(A) \stackrel{\text{def}}{=} S f(D)$~~

$f(A) = S f(D) S^{-1}$ ← entrywise, $D = \mathrm{diag}(\lambda_i)$

Again <u>if A is diagonalizable</u>, i.e. $A = SDS^{-1}$, then

$$e^A = Se^DS^{-1} = S\begin{pmatrix}e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n}\end{pmatrix}S^{-1} \quad \text{etc.}$$

- <u>Example</u>. $A = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$. [One can check: $\begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&x\end{pmatrix}\begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix}x&x&x&x\\x&x&x&x\\x&x&x&x\\x&x&x&x\end{pmatrix}$]

$$\Rightarrow A^{100} = \begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&4^{100}\end{pmatrix}\begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}^{-1} = \begin{pmatrix}4^{99}&4^{99}&4^{99}&4^{99}\\4^{99}&4^{99}&4^{99}&4^{99}\\4^{99}&4^{99}&4^{99}&4^{99}\\4^{99}&4^{99}&4^{99}&4^{99}\end{pmatrix} = 4^{99}A.$$

$$e^A = \begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&e^4\end{pmatrix}\begin{pmatrix}1&1&1&1\\-1&0&0&1\\0&-1&0&1\\0&0&-1&1\end{pmatrix}^{-1} = \frac{\pi}{4}A$$

Ex. $A = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

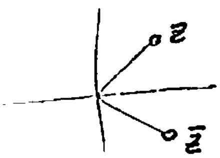
$$e^A \approx \begin{bmatrix}9.36 & 6.36 & 6.36\\ 6.36 & 6.36 & 6.36\\ 6.36 & 6.36 & 6.36\end{bmatrix}$$

3.

Complex Vectors and matrices. Inner product.

- Recall: complex conjugate $z = a+bi$, $\bar{z} = a-bi$.

  Prop: (a) $z \cdot \bar{z} = a^2+b^2 = |z|$
  (b) $\overline{xz} = \bar{x}\,\bar{z}$



$\mathbb{C}^n \supset \mathbb{R}^n$.

- **Def** Inner product on $\mathbb{C}^n$ is defined as

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i \bar{v}_i = \vec{u}^T \cdot \overline{\vec{v}}$$

where $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Properties

(a) If $\vec{u}, \vec{v} \in \mathbb{R}^n \Rightarrow \langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$ (dot product).

(b) $\langle \vec{u}, \vec{u} \rangle = \sum u_i \bar{u}_i = \sum |u_i|^2 = \|u\|^2$.

Warning: $\vec{u} \circ \vec{u}$ may be complex, $\neq \|u\|^2$

That's why conjugation is needed in def of $\langle \vec{u}, \vec{v} \rangle$.

(c) $\langle \lambda \vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle$; $\langle \vec{u}, \mu \vec{v} \rangle = \bar{\mu} \langle \vec{u}, \vec{v} \rangle$.

Ex $\vec{u} = \begin{bmatrix} i \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2+i \\ 3-i \end{bmatrix}$. $\vec{u} \cdot \vec{v} = i(2-i) + 1(3+i) = 1+2i+3+i = 4+3i$.

$\|\vec{v}\| = \sqrt{(2^2+1^2)+(3^2+1^2)} = \sqrt{15}$.

A: $n \times m$, entries $\in \mathbb{C}$.

Def The conjugate transpose $A^* = \overline{A^T} = (\bar{A})^T$.

Ex $A = \begin{bmatrix} 1+i & 2 \\ 3-2i & 7i \end{bmatrix}$ $\quad A^* = \begin{bmatrix} 1-i & 3+2i \\ 2 & -7i \end{bmatrix}$

Properties:

(a) A has real entries $\Rightarrow A^* = A^T$.

(b) $(A+B)^* = A^* + B^*$

(c) $(kA)^* = \bar{k} A^*$

(d) $(AB)^* = B^* A^*$

(e) $(A^*)^* = A$

(f) $\boxed{\langle A\vec{u}, \vec{v} \rangle = \langle \vec{u}, A^*\vec{v} \rangle}$ $\Rightarrow$ (For real $A, \vec{u}, \vec{v}$: $\langle A\vec{u}, \vec{v} \rangle = \langle \vec{u}, A^T\vec{v} \rangle$.)

$\langle A\vec{u}, \vec{v} \rangle = (A\vec{u})^T \cdot \overline{\vec{v}} = \vec{u}^T A^T \overline{\vec{v}} = \vec{u}^T \overline{A^* \vec{v}} = \vec{u}^T \overline{A^* \vec{v}}$

$= \langle \vec{u}, A^* \vec{v} \rangle$.