8.1. Symmetric matrices.

lec 22
04/09

Def A real matrix A is called <u>symmetric</u> if $A^T = A$.
A complex matrix A is called <u>Hermitian</u> if $A^* = A$.

Ex $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is symmetric, $\begin{bmatrix} 5 & 1-i \\ 1+i & 4 \end{bmatrix}$ is Hermitian.

<u>Note</u>: • diag. entries must be real!
• A real symmetric matrix is Hermitian, too.
• A Hermitian $\Rightarrow$ $\langle A\vec{u}, \vec{v}\rangle = \langle \vec{u}, A\vec{v}\rangle$ $\forall \vec{u}, \vec{v}$ (by Prop. (f) p.90) $(*)$

Thm Suppose A is Hermitian. (or real symmetric) Then:
(a) All eigenvalues of A are <u>real</u>;
(b) Eigenvectors of A corresponding to different eigenvalues are <u>orthogonal</u>

(a) Let $A\vec{v} = \lambda\vec{v}$, $A = A^*$.

$$\langle A\vec{v}, \vec{v}\rangle = \langle \lambda\vec{v}, \vec{v}\rangle = \lambda\langle \vec{v}, \vec{v}\rangle = \lambda\|\vec{v}\|^2$$

$$= \langle \vec{v}, \underset{\substack{\| \\ A\vec{v} = \lambda\vec{v}}}{A^*\vec{v}}\rangle = \langle \vec{v}, \lambda\vec{v}\rangle = \bar{\lambda}\langle \vec{v}, \vec{v}\rangle = \bar{\lambda}\|\vec{v}\|^2.$$

Since $\|\vec{v}\| \neq 0$ it follows that $\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

(b) Let $A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\lambda_1 \neq \lambda_2$

Consider $\langle A\vec{v}_1, \vec{v}_2\rangle = \langle \lambda_1\vec{v}_1, \vec{v}_2\rangle = \lambda_1\langle \vec{v}_1, \vec{v}_2\rangle$

$= \langle \vec{v}_1, A\vec{v}_2\rangle$ (since A is Hermitian, using $(*)$)

$= \langle \vec{v}_1, \lambda_2\vec{v}_2\rangle = \bar{\lambda}_2\langle \vec{v}_1, \vec{v}_2\rangle$

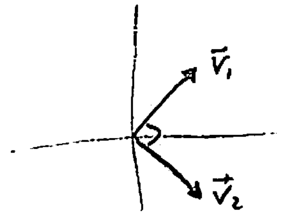
$= \lambda_2\langle \vec{v}_1, \vec{v}_2\rangle$ (since λ_2 is real, by part (a))

Comparing & $(\lambda_1 \neq \lambda_2) \Rightarrow \langle \vec{v}_1, \vec{v}_2\rangle = 0$.

Ex $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ (from previous classes)

$\lambda_1 = 4, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\lambda_2 = 2, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Not only $\exists$ basis of eigenvectors but an orthonormal basis:

$$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

$$\vec{v}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

Diagonalization. $A = S^{-1} D S$, $\quad S^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $\quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

$\uparrow$
orthogonal matrix $\Rightarrow$ $\quad S = (S^{-1})^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

· Toward Spectral Thm.

Def A subspace E is called invariant for A if

$$Ax \in E \quad \forall x \in E$$

Ex (a) $A = \text{proj}_V \Rightarrow V, V^\perp$ invariant

(b) Same for $A = \text{ref}_V$

(c) $E = \mathbb{R}^n$; $E = \{0\}$ is invariant $\forall A$

(d) $A =$ rotation in $\mathbb{R}^2$ $\Rightarrow$ no invariant subspaces except $\{\vec{0}\}$, $\mathbb{R}^2$

(e) $\text{span}\{\vec{v}\}$ $\forall$ eigenvalue

(f) ~~$E_\lambda = \ker$~~ $\forall$ eigenspace E_λ.

· **Prop** Let A be Hermitian. If E is invariant then $E^\perp$ is invariant

Let $\vec{x} \in E^\perp$: $A\vec{x} \in E^\perp$? $\Leftrightarrow \langle A\vec{x}, \vec{e} \rangle = 0 \quad \forall \vec{e} \in E$?

$$\langle A\vec{x}, \vec{e} \rangle = \langle \underset{E^\perp}{\underset{\cap}{x}}, \underset{E}{\underset{\cap}{A\vec{e}}} \rangle = 0$$

Spectral Thm (8.1) Let A be a Hermitian matrix.

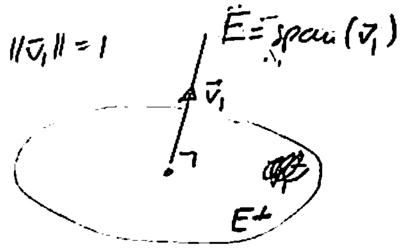
Then there exists an orthogonal basis of eigenvectors.

Consequently, A can be orthogonally diagonalized:

$$A = S^{-1} D S, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

Both $S, S^{-1} = S^T$ are orthogonal.

~~Idea~~ • Find one eigenvector: $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $\|\vec{v}_1\| = 1$



$E := \mathrm{span}(\vec{v}_1)$ is invariant

$\Rightarrow E^\perp$ is invariant (by Prop.)

• Consider the "restriction of A onto $E^\perp$". (in dim $n-1$)

$B : E^\perp \to E^\perp$ the linear transf: $Bx = x$ $(\forall x \in E^\perp)$

Find an eigenvector $\vec{v}_2$ ($\|\vec{v}_2\| = 1$) of B; it is also an eigenvector of A

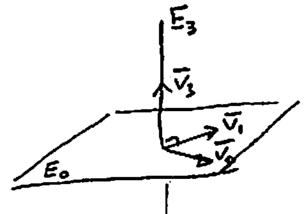
.....

$\longrightarrow$ find n orthonormal eigenvectors of A.

Ex $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 3$.

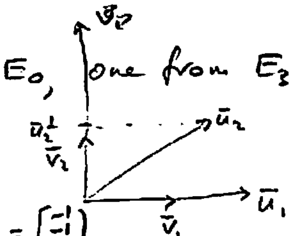
$\lambda = 0$: $E_0 = \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$\lambda = 3$: $E_3 = \mathrm{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.



Note: $E_0 \perp E_3$ in accordance with Thm p.91.

Orthogonal basis of eigenvectors: choose two from E_0, one from E_3



Gram-Schmidt on $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\quad \vec{u}_2^\perp = \vec{u}_2 - (\vec{v}_1 \cdot \vec{u}_2)\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$\vec{v}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$\vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{A = S^{-1} D S}$, $\quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\quad S^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \leftarrow$ orthogonal. $\quad S = (S^{-1})^T$.

~~Geometric: A = projection onto span~~ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$~~, followed by scaling~~ in the direction of ~~$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by factor 3~~.

Spectral decomposition

Why is orthonormal basis of eigenvectors is good? Coeff's can be easily computed:

Recall: If $\vec{v}_1, \dots, \vec{v}_n$ is an orthonormal basis of $\mathbb{R}^n$, then

$$\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$$

("Basis decomposition" Thm 5.1.6)

$$\vec{x} = \vec{v}_1(\vec{v}_1^T \vec{x}) + \dots + \vec{v}_n(\vec{v}_n^T \vec{x})$$
$$= (v_1 v_1^T + \dots + \vec{v}_n \vec{v}_n^T)\vec{x}.$$

Hence

$$\boxed{I_n = v_1 v_1^T + \dots + v_n v_n^T}$$

Multiply by A:

$$A = A\vec{v}_1\vec{v}_1^T + \dots + \vec{v}_n\vec{v}_n^T$$
$$= \lambda_1 \vec{v}_1\vec{v}_1^T + \dots + \lambda_n \vec{v}_n \vec{v}_n^T$$

$\Rightarrow$

Thm (Spectral decomposition) Let A be a symmetric matrix. Then A can be expressed as

$$\boxed{A = \sum_{i=1}^{n} \lambda_i v_i v_i^T}$$

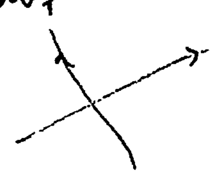
where λ_i are eigs (counting multiplicities) and $\{\vec{v}_i\}$ be corresponding ~~basis of~~ orthonormal basis of eigenvectors.

Remarks

1. Same for Hermitian A: $A = \sum_{i=1}^{n} \lambda_i v_i v_i^*$

2. Recall: $v_i v_i^T$ is the orthogonal proj. onto span(v_i) ← line
   – rank one proj.

   Hence spectral decomposition expresses A as a linear combination of rank-one projections



Ex (a) (p.92): $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$A = 4\bar{v}_1\bar{v}_1^T + 2\bar{v}_2 v_2^T \quad , \quad \bar{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

(b) (p. 93) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A = 3\underbrace{\bar{v}_3\bar{v}_3^T}_{\cancel{3\times\text{orth. proj.}}} \quad , \quad \bar{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (\text{Check}).$$

Geometry: A = orth. projection onto span $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ followed by scaling in that direction by factor 3.