Math 214 (Winter '08)

Midterm 1

February 7, 2008

Time: 90 minutes

1 (10 points). Solve the linear system:

$$\begin{aligned} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 4 \\ 7x + 8y + 9z &= 9 \end{aligned}$$

(for full credit you need to use the matrix form).

2 (10 points). Which of the following sets are a linear subspace (explain your answer):

i)

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} : x + y = z + t + 1 \right\}$$

ii)

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} : x + y = z + t \right\}$$

iii)

$$W = \{(x, y) \in \mathbb{R}^2 : x = y^2\}$$

iv)

$$W = \left\{ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$$

v) W is the set of polynomials of degree at most 2.

3 (15 points). For which values of k is the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & k \end{bmatrix}$$

invertible. If A is invertible, explain how to find the inverse of A and illustrate a few steps of the computation (you don't need to do the complete computation of A^{-1}).

4 (10 points). For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

describe the images and kernels of the matrices A, A^2, and A^3 geometrically.

5 (20 points). Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 3 \\ 4 & 9 & 14 & 11 \end{bmatrix}$$

a) What are the dimensions of $\mathrm{Im}(T)$ and $\mathrm{Ker}(T)$? What relation do they satisfy?

b) Find a basis of $\mathrm{Im}(T)$.

c) Find a basis of $\mathrm{Ker}(T)$.

6 (10 points). Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$2x_1 - x_2 + 2x_3 + 4x_4 = 0$$

Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $\mathrm{Ket}(T) = 0$ and $\mathrm{Im}(T) = V$. Describe T by its matrix A. Explain why $\mathrm{Ket}(T) = 0$.

7 (15 points).

i) Find a basis for the linear span of the vectors $\vec{w}_1, \ldots \vec{w}_5$, where

$$\vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_4 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \text{ and } \quad \vec{w}_5 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

ii) Give a basis of $\mathbb{R}^4$ that contains the vectors $\vec{v}_1$ and $\vec{v}_2$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

8 (10 points). Suppose that X is 2×2 matrix satisfying $AXB = C$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Find X.