Math 419 (Fall '08)

Midterm 1

October 7, 2008

Time: 80 minutes

1 (10 points). Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What about into

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}?$$

Explain.

2 (10 points) Find a 3×3 matrix A such that

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 0 \\ k \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

where

i) $k = 1$;

ii) $k = 0$.

What is the (qualitative) difference?

3 (10 points) Find the inverse of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

4 (20 points). Let V be the subspace of $\mathbb{R}^5$ defined by the equations

$$\begin{aligned} 2x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ x_1 + x_3 + 2x_4 &= 0 \end{aligned}$$

i) Show that V is a linear subspace.

ii) Give a linear transformation S such that $V = \mathrm{Ker} S$.

iii) Give a linear transformation T such that $V = \mathrm{Im} T$.

iv) Find a basis of V.

5 (15 points). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation associated to the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$$

a) What is n and m? What are the dimensions of $\text{Im}(T)$ and $\text{Ker}(T)$? What relation do they satisfy?

b) Find a basis of $\text{Im}(T)$.

c) Find a basis of $\text{Ker}(T)$.

6 (10 points). Let

$$V = \text{Span}\left(\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}\right).$$

Find a basis of V that contains the vectors:

$$\begin{bmatrix}2\\2\\4\\4\\6\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\2\\2\\4\\4\end{bmatrix}.$$

7 (15 points). Let P_2 be the set of polynomials of degree at most 2 and M_2 be the set of 2×2 matrices.

1. Explain why P_2 and M_2 are abstract linear spaces.
2. Which of the following subsets of P_2 or M_2 are linear subspaces:
   i) $V = \{p \in P_2 \mid p(0) = 2\}$;
   ii) $V = \{p \in P_2 \mid p(t) - t \cdot p'(t) = 0\}$ (where p' denotes the derivative);
   iii) $V = \{p \in P_2 \mid p(t) = p(-t) \text{ for all } t\}$;
   iv) $V = \{A \in M_2 \mid A \text{ is invertible}\}$;
   v) $V = \{A \in M_2 \mid \operatorname{tr}(A) = 0\}$, where the trace of A, $\operatorname{tr}(A)$, is the sum of the elements on the diagonal.

   Explain!

8 (10 points). True or False:

1. There exists a 3×4 matrix with rank 4.
2. If A is in rref then at least one of the entries in each column must be 1.
3. If A is a 4×4 matrix and the system

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

has a unique solution, then the system

$$A\vec{x} = \vec{0}$$

has only the solution $\vec{x} = \vec{0}$.

4. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.
5. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$.
6. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
7. The column vectors of a 5×4 matrix must be linearly dependent.
8. The column vectors of a 4×5 matrix must be linearly dependent.
   Explain 7 and 8!

9. There exists a 3×3 matrix A such that $\text{Im}(A) = \ker(A)$.
10. The kernel of an abstract linear transformation is a linear subspace of the domain.