1 (15 pts.) Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No reasoning is required.)

(T) For any two $n \times n$ matrices A and B, if AB is invertible, then both A and B are invertible.

(T) If $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ are both consistent systems, so is $A\vec{x} = \vec{b} + \vec{c}$.

(F) Let A, B and C be three square matrices, and $AB = AC$, then $B = C$.

(F) Let L be a staight line in $\mathbb{R}^2$ passing the origin. Then the transformation "orthogonal projection onto L" is the only linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose image is L.

(T) Let $\vec{v}_1$ and $\vec{v}_2$ be linearly independent vectors, and let $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$, $\vec{v}_4 = \vec{v}_1 - 2\vec{v}_2$. Then any two vectors among v_1, v_2, v_3, v_4 are linearly independent.

2 (15 pts.) Solve the following system of linear equations

$$\begin{cases} x + y + z + w = 1 \\ 2x + 2y + 2z + 3w = 4 \\ 2x + 3y + 4z + 5w = 6 \\ x + 3y + 5z + 10w = 15 \end{cases}$$

(No partial credit will be given for this problem. Check your answer carefully.)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 10 & 15 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 9 & 14 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 2 & 4 & 0 & -4 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

free

So solution is $\begin{cases} x = t+1 \\ y = -2t-2 \\ z = t \\ w = 2 \end{cases}$

Check: $(t+1)+(-2t-2)+t+2 = 1$

$2(t+1)+2(-2t-2)+2t+6 = 4$

$2(t+1)+3(-2t-2)+4t+10 = 6$

$(t+1)+3(-2t-2)+5t+20 = 15$

OK.

3 (15 pts.) Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 5 & 1 \\ 2 & 6 & 2 \end{pmatrix}$$

(a) Is A invertible? If yes, calculate A^{-1}.

(b) Find the matrix X so that

$$AX = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

(No partial credit will be given for this problem. Check your answer carefully.)

(a)
$$\left(\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 & 1 & 0 \\ 2 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 1 & 5 & 1 & 0 & 1 & 0 \\ 2 & 6 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & -1 & -\frac{1}{2} & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{7} & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & -\frac{8}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{array}\right) \to \begin{pmatrix} 1 & 0 & \frac{17}{7} & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{4} & -\frac{7}{8} \end{pmatrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{9}{4} & \frac{17}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{4} & -\frac{7}{8} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{9}{4} & \frac{17}{8} \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{7}{8} \end{pmatrix} \qquad (\text{Check} : A^{-1}A = I_3)$$

(b)
$$X = A^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -4 & -18 & 17 \\ 0 & 4 & -2 \\ 4 & 6 & -7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -5 & 11 & -5 \\ 2 & 2 & 2 \\ 3 & -5 & 3 \end{pmatrix}$$

4 (15 pts.) Let T_1, T_2 be the linear transformations

$$T_1 : \text{Reflection with respect to the line } y = x;$$

$$T_2 : \text{Rotation couterclockwise by 45 degree.}$$

Denote the matrix of T_1 by A, and the matrix of T_2 by B.

(a) Write down A and B explicitly.

(b) Calculate the matrix ABA.

(c) Describe the linear transformation $T_1 \circ T_2 \circ T_1$ as a single geometric transformation.

(a). $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

(b) $ABA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

(c) Since $ABA = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix}$

we see $T_1 \circ T_2 \circ T_1$ = Rotation clockwise by 45°.

5 (20 pts.) We can identify any 2×2 matrix X with a 4-vector $\vec{x}$ in $\mathbb{R}^4$ as follows:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

By this way, we can identify the set of all 2×2 matrices with the 4-dimensional space $\mathbb{R}^4$, and thus convert many operations on matrices as operations on $\mathbb{R}^4$. For example, we have the following transformations

$$T_1(\vec{x}) := \det(X) = ad - bc,$$

$$T_2(\vec{x}) := \text{ the vector corresponding to } X^{-1} \text{ if } X \text{ is invertible,}$$

$$T_3(\vec{x}) := \text{ the vector corresponding to } AX, \text{ where } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Among T_1, T_2, T_3, which one is linear? Write down its matrix B.

(b) Is the matrix B invertible? If yes, find its inverse.

(c) We could also identify any 2×2 matrix X with a 4-vector $\vec{y}$ in $\mathbb{R}^4$ as follows:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff \vec{y} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}.$$

In this case, we denote the matrix for the linear transformation you found in part (a) by C. Figure out the relation between B and C.

(a) T_3 is linear, with matrix $B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$ since $T_3\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+2c \\ b+2d \\ 3a+4c \\ 3b+4d \end{pmatrix}$

(b) Since T_3 is invertible, with $T_3^{-1}(\vec{x}) = A^{-1}\vec{x} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\vec{x} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}\vec{x} = \frac{1}{2}\begin{pmatrix} -4a+2c \\ -4b+2d \\ 3a-c \\ 3b-d \end{pmatrix}$

So B is invertible with $B^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 0 & 2 & 0 \\ 0 & -4 & 0 & 2 \\ 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}$

(c) B and C are similar, since they are matrices for the same linear transformation, under different basis.
In fact, $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} B \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$

6 (20 pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 7 & 7 \\ 2 & 1 & 9 & 10 \\ 2 & 0 & 4 & 6 \end{pmatrix},$$

(a) Find a basis for the image of A.

(b) Find a basis for the kernel of A.

(c) For which value(s) of the constant a is the vector $\begin{pmatrix} a \\ 2 \\ 4 \end{pmatrix}$ a linear combination of the column vectors of A?

RREF(A): $\begin{pmatrix} 1 & 1 & 7 & 7 \\ 2 & 1 & 9 & 10 \\ 2 & 0 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & 7 \\ 0 & -1 & -5 & -4 \\ 0 & -2 & -10 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & 7 \\ 0 & 1 & 5 & 4 \\ 0 & -2 & -10 & 8 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(a) Basis of Im(A): $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $Ker(A) = \begin{pmatrix} -2t-3s \\ -5t-4s \\ t \\ s \end{pmatrix} \Rightarrow$ Basis $\begin{pmatrix} -2 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -4 \\ 0 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 7 & 7 & a \\ 2 & 1 & 9 & 10 & 2 \\ 2 & 0 & 4 & 6 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & 7 & a \\ 0 & -1 & -5 & -4 & 2-2a \\ 0 & -2 & -10 & -8 & 4-2a \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 7 & 7 & a \\ 0 & 1 & 5 & 4 & 2a-2 \\ 0 & -2 & -10 & -8 & 4-2a \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 2 & 3 & 2-a \\ 0 & 1 & 5 & 4 & 2a-2 \\ 0 & 0 & 0 & 0 & 2a \end{pmatrix}$

$\therefore \begin{pmatrix} a \\ 2 \\ 4 \end{pmatrix}$ is a linear combination $\Leftrightarrow 2a = 0 \Leftrightarrow a = 0$.

In this case, $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.