1 (15 pts.) Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No reasoning is required.)

(T) Let A be an $n \times n$ matrices so that $A + A^{2012}$ is invertible. Then A is invertible.

$A + A^{2012} = A(I + A^{2011})$

(T) If a system of linear equations has two different solutions, it must has infinitely many solutions.

(T) There exists a 2×2 matrix A such that $A^2 = -I_2$.

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (rotation by $\frac{\pi}{2}$)

(F) The composition of two reflections is still a reflection.

rotation

(F) If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, then $\vec{v}_3$ must be a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

It may happens that $\vec{v}_1 \parallel \vec{v}_2$ and $\vec{v}_3$ is independent

2 (15 pts.) Solve the system of linear equations

$$\begin{cases} x + 2y + z + w = 2, \\ 2x + 2y \phantom{{}-3z} + 2w = 0, \\ 4x + 2y - 3z + 5w = -2. \end{cases}$$

(No partial credit will be given for this problem. Check your answer carefully.)

Sol.

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 0 & 2 & 0 \\ 4 & 2 & -3 & 5 & -2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & -2 & -2 & 0 & -4 \\ 0 & -6 & -7 & 1 & -10 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 6 & 7 & -1 & 10 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 & -2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & \underset{\substack{\uparrow \\ w=t}}{-1} & -2 \end{array}\right) \Rightarrow \begin{cases} x = -4 \\ y = -t+4 \\ z = t-2 \\ w = t \end{cases}$$

Check:

$-4 - 2t + 8 + t - 2 + t = 2$ ✓

$-8 - 2t + 8 \qquad + 2t = 0$ ✓

$-16 - 2t + 8 - 3t + 6 + 5t = -2$ ✓

3 (15 pts.) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

(a) Find A^{-1}.

(b) Using part (a) to solve the linear system $A\vec{x} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$.

(No partial credit will be given for this problem. Check your answer carefully.)

Sol: (a)
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 3 & 1 & 2 & -1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 3 \end{array}\right). \Rightarrow A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ -4 & -1 & 3 \end{pmatrix}$$

Check: $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ -4 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1+4-4 & 1+0-1 & -1-2+3 \\ 2+2-4 & 2+0-1 & -2-1+3 \\ 2+6-8 & 2+0-2 & -2-3+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

(b) $\vec{x} = A^{-1}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ -4 & -1 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ -13 \end{pmatrix}.$

Check: $A\vec{x} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ -13 \end{pmatrix} = \begin{pmatrix} 6+10-13 \\ 12+5-13 \\ 12+15-26 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ ✓.

4 (15 pts.) Consider the matrix

$$A = \begin{array}{l} \left(\begin{array}{cccc} 1 & 0 & 0 & 4 \\ 2 & 0 & 1 & 2 \\ m & 4 & 1 & 9 \end{array}\right) \end{array} \begin{array}{l} \leftarrow \text{month/date} \\ \leftarrow \text{year} \\ \leftarrow \text{course number} \end{array}$$

where m is an arbitrary constant.

(a) What is the rank of A?

(b) Find a basis for the image of A.

(c) Find a basis for the kernel of A.

(Your solution may contains the constant m.)

<u>Sol</u>:

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 1 & 2 \\ m & 4 & 1 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -6 \\ 0 & 4 & 1 & 9-4m \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -6 \\ 0 & 4 & 0 & 15-4m \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & -6 \\ 0 & 1 & 0 & \frac{15}{4}-m \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{15}{4}-m \\ 0 & 0 & 1 & -6 \end{pmatrix}$$

(a) rank $A = 3$

(b) $\begin{pmatrix} 1 \\ 2 \\ m \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ [or: $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$: obviously linearly independent!]

(c) $A\vec{x} = 0 \Rightarrow \vec{x} = \begin{pmatrix} -4t \\ (\frac{15}{4}-m)t \\ 6t \\ t \end{pmatrix} = t \begin{pmatrix} -4 \\ m-\frac{15}{4} \\ 6 \\ 1 \end{pmatrix}$

$\therefore$ basis of Ker A: $\begin{pmatrix} -4 \\ m-\frac{15}{4} \\ 6 \\ 1 \end{pmatrix}$.

(Workspace for problem 5)

Sol. (a) $T_1\begin{pmatrix}x\\y\\z\end{pmatrix} = x+2y+3z \Rightarrow A_1 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}$

$T_2\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2y\\2x\\0\end{pmatrix} \Rightarrow A_2 = \begin{pmatrix}0 & -2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

(b) $A_3 = \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}0 & -2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}4 & -2 & 0\end{pmatrix}$

(c) $T_4\begin{pmatrix}1\\0\\0\end{pmatrix}$ = rotation of projection of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ = rotation of $\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$T_4\begin{pmatrix}0\\1\\0\end{pmatrix} = \text{- - - - - - - - - -} \begin{pmatrix}0\\1\\0\end{pmatrix} = \text{. . . . . .} \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$

$T_4\begin{pmatrix}0\\0\\1\end{pmatrix} = \text{. . . - - - - - - -} \begin{pmatrix}0\\0\\1\end{pmatrix} = \text{- - - - - -} \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\therefore A_4 = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.

(d) Since $A_2 = 2A_4 = \begin{pmatrix}2 & & \\ & 2 & \\ & & 2\end{pmatrix} A_4$, we see:

T_4 = First project $\begin{pmatrix}x\\y\\z\end{pmatrix}$ onto xy plane,
then rotate counterclockwise by $\frac{\pi}{2}$,
then scale by 2.

[In general: $\begin{pmatrix}x\\y\\z\end{pmatrix} \times \begin{pmatrix}u\\v\\w\end{pmatrix}$ = first project $\begin{pmatrix}x\\y\\z\end{pmatrix}$ onto the plane $ux+vy+wz=0$,
then rotate by $\frac{\pi}{2}$ counterclockwise (looking from the direction of $\begin{pmatrix}u\\v\\w\end{pmatrix}$),
then scale by $\left|\begin{pmatrix}u\\v\\w\end{pmatrix}\right|$.]

6 (20 pts.) Let A be an $m \times n$ matrix so that

$$A\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix},\quad A\begin{pmatrix}0\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\quad A\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}.$$

(a) What is n? What is m?

(b) Show that the vectors $\vec{v}_1=\begin{pmatrix}1\\-2\\0\end{pmatrix}, \vec{v}_2=\begin{pmatrix}0\\2\\1\end{pmatrix}, \vec{v}_3=\begin{pmatrix}2\\0\\0\end{pmatrix}$ form a basis of $\mathbb{R}^3$.

(c) Express the vector $\vec{w}_1=\begin{pmatrix}1\\2\\3\end{pmatrix}$ as a linear combination of the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

(d) Find $A\vec{w}_1$.

(e) Let $\vec{w}_2$ be a vector in $\mathbb{R}^3$ whose coordinate vector is $[\vec{w}_2]_{\mathcal{B}}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ under the basis $\mathcal{B}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$. Find $\vec{w}_2$ and $A\vec{w}_2$.

Sol. (a) $m=2$, $n=3$

(b) $$\begin{pmatrix}1&0&2\\-2&2&0\\0&1&0\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&2&4\\0&1&0\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&1&0\\0&2&4\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&1&0\\0&0&4\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&1&0\\0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}1\\-2\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}$ linearly independent $\Rightarrow$ basis.

(c) $$\left(\begin{array}{ccc|c}1&0&2&1\\-2&2&0&2\\0&1&0&3\end{array}\right)\to\begin{pmatrix}1&0&2&1\\0&2&4&4\\0&1&0&3\end{pmatrix}\to\begin{pmatrix}1&0&2&1\\0&1&0&3\\0&2&4&4\end{pmatrix}\to\begin{pmatrix}1&0&2&1\\0&1&0&3\\0&0&4&-2\end{pmatrix}\to\begin{pmatrix}1&0&2&1\\0&1&0&3\\0&0&1&-\frac{1}{2}\end{pmatrix}$$

$$\to\begin{pmatrix}1&0&0&2\\0&1&0&3\\0&0&1&-\frac{1}{2}\end{pmatrix}\quad\Rightarrow \vec{w}_1=2\vec{v}_1+3\vec{v}_2-\tfrac{1}{2}\vec{v}_3$$

(d) $A\vec{w}_1=2A\vec{v}_1+3A\vec{v}_2-\frac{1}{2}A\vec{v}_3=2\begin{pmatrix}2\\4\end{pmatrix}+3\begin{pmatrix}0\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}7/2\\7\end{pmatrix}$

(e) $\vec{w}_2=\vec{v}_1+2\vec{v}_2+3\vec{v}_3=\begin{pmatrix}1\\-2\\0\end{pmatrix}+2\begin{pmatrix}0\\2\\1\end{pmatrix}+3\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}7\\2\\2\end{pmatrix}$

$A\vec{w}_2=A\vec{v}_1+2A\vec{v}_2+3A\vec{v}_3=\begin{pmatrix}2\\4\end{pmatrix}+2\begin{pmatrix}0\\0\end{pmatrix}+3\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}$