Math 419 – Linear Spaces and Matrix Theory

Exam 2 11/06/2012

▶ **Your PRINTED name is: Practice Exam**

Grading
1
2
3
4
5
6
Total:

▶ **This is a 80-minutes closed-book exam. No calculator or notes are allowed.**

▶ **This exam booklet contains six problems, on 14 sheets of paper including the front cover.**

▶ **The last two pages are to be used as scratch paper. Please detach them carefully before exam. DO NOT write your solution on the scratch paper!**

▶ **Full credit will be given only if all reasoning and work is provided. For some problems, you may get partial credits if you show your idea clearly.**

1 (15 pts.) Which of the following statements are true? Put a (T) before the correct ones and an (F) before the wrong ones. (No justification is required.)

(   ) If A is the matrix of an orthogonal projection, then $A^2 = A$.

(   ) If A and B are both orthogonal $n \times n$ matrices, then $AB = BA$.

(   ) The set of all smooth functions $f(x)$ such that $\int_{-1}^{1} f(x)dx = 0$ is a linear space.

(   ) The map $T(f) = \begin{pmatrix} f(0) & f(1) \\ f(2) & f(3) \end{pmatrix}$ is an isomorphism from P_4 to $\mathbb{R}^{2\times 2}$.

(   ) If the entries of two vectors $\vec{v}$ and $\vec{w}$ are all negative, then the angle between $\vec{v}$ and $\vec{w}$ must be an acute angle.

2 (15 pts.) Fill in the blanks. (No justification is required. **No partial credit.**)

(a) The trace of the matrix $\begin{pmatrix} 3 & 4 & 5 \\ 12 & 3 & 14 \\ 0 & 0 & 1 \end{pmatrix}$ is ______________.

(b) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be orthonormal vectors in $\mathbb{R}^{15}$. Then the length of the vector $v = \vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 - 2\vec{v}_4$ is ______________.

(c) The dimension of the space of "all polynomials in $\mathcal{P}_3$ that are also odd functions" is ______________.

(d) Under the basis

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

of $\mathbb{R}^{2\times 2}$, the coordinate vector of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is ________.

(e) Write down a 4×4 orthogonal matrix that is not the identity matrix

______________________________.

3 (15 pts.) Let V be the subset of $\mathbb{R}^{3\times 3}$ consisting of all 3×3 matrices A such that $A^T = -A$.

(a) Argue that V is a linear subspace of P_3.

(b) Find a basis of V.

(c) Consider the linear transformation $T : V \to V$ defined by

$$T(A) = S^T AS,$$

where S is the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & -1 & 0 \end{pmatrix}$. Find the matrix of T with respect to the basis you get in part (b).

(Extra workspace for problem 3)

4 (15 pts.)

(a) Find the least square solution to the system $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$.

(b) Find a linear function of the form $f(t) = c_0 + c_1 t$ that best fits the data $(0,1), (1,2), (2,3), (3,5)$.

(c) Find the matrix of the orthogonal projection onto the subspace of $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(Extra workspace for problem 4)

5 (20 pts.) Let $V = \text{Im}(A)$, where $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$.

(a) Find an orthonormal basis of V.

(b) Find the QR decomposition of A.

(c) Find the orthogonal projection of $\vec{v} = (1\ 2\ 3\ 4)^T$ onto V.

(d) Find an orthonormal basis of $V^{\perp}$.

(Workspace for problem 5)

(Extra workspace for problem 5)

6 (20 pts.) Let V be an inner product space, and X_1, X_2, X_3 three elements in V. Suppose we know that $\langle X_i, X_j \rangle$ is the entry a_{ij} of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}.$$

Use this information to answer the following questions.

(a) Find $|X_1|$.

(b) Find the angle enclosed by X_2 and X_3.

(c) Find $|X_1 + X_2|$.

(d) Find $\text{Proj}_V(X_3)$, where $V = \text{span}(X_1, X_2)$.

For each of the following matrices B, there is no way to find elements X_1, X_2, X_3 so that the entry b_{ij} is $\langle X_i, X_j \rangle$. Explain why (case by case).

(e) $B = \begin{pmatrix} -1 & 3 & 0 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 3 & 0 \\ 3 & 5 & 8 \\ 0 & 8 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 3 & 1 \\ 3 & 50 & 8 \\ 0 & 8 & 4 \end{pmatrix}$

(Extra workspace for problem 6)

The page is intended for use as scratch paper.

The page is intended for use as scratch paper.