Midterm Exam

SOLUTIONS

Total . 124

Math 419, Prof. Roman Vershynin
Winter 2013

Name: ____________________

Read the following information before starting the exam:

- No calculators, computers, phones or any other computing and communication devices are allowed on the exam. Books and notes are not allowed. You may use one letter-size handwritten sheet. Only write on one side of that sheet. No photocopies of anything are allowed.
- Show all work, clearly and in order, if you want to get full credit. Points will be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.
- This work is strictly individual.

1. (30 *points*) Determine whether the following statements are **true or false**. Justify your answer.

a. (~~3~~5 *pts*) There exists a system of three linear equations with three unknowns that has exactly three solutions.

False : any linear system may have 0, 1, or ∞ solutions

b. (~~3~~5 *pts*) A system that has more unknowns than equations always has a solution.

False :
$$\begin{cases} x+y+z=0 \\ x+y+z=1 \end{cases}$$
has no solutions.

c. (~~3~~5 *pts*) There exists an invertible matrix A such that $A^2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

False : For any invertible A, A^2 is invertible.
But $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible (cols are linearly dependent).

d. (*5 pts*) The vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}$ form a basis of $\mathbb{R}^2$.

False : a basis of R^2 always consists of 2 vectors.

e. (*5 pts*) Any set of vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ has a subset which forms a basis of $\mathbb{R}^n$.

True : removing redundant vectors from this set leads to a linear independent subset, which spans the same subspace ($\mathbb{R}^n$)

f. (*5 pts*) There are infinitely many 3-dimensional subspaces of $\mathbb{R}^4$.

True a rotation of any 3-dimensional subspace is a subspace. There are ∞ rotations.

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2. ~~B~~. (~~??~~ *points*) Determine the values of k for which the following linear system has no solutions, a unique solution, and infinitely many solutions, respectively:

$$\begin{cases} x - y = k \\ x - ky = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & -1 & k \\ 1 & -k & 1 \end{array}\right] \xrightarrow[-I]{} \left[\begin{array}{cc|c} 1 & -1 & k \\ 0 & 1-k & 1-k \end{array}\right]$$

If $k = 1$ then rref is $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x - y = 1$.

∞ solutions.

If $k \neq 1$ then $\left[\begin{array}{cc|c} 1 & -1 & k \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{+I} \left[\begin{array}{cc|c} 1 & 0 & k+1 \\ 0 & 1 & 1 \end{array}\right]$. $\begin{cases} x = k+1 \\ y = 1 \end{cases}$

1 solution.

Answer: for $k = 1$, there are infinitely many solutions
for any $k \neq 1$, there is a unique solution.

3 ~~4~~. (15 *points*) Find the dimensions of the image and of the kernel for the following matrices:

a. (5 *pts*) $A = \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$

Columns are linearly independent $\Rightarrow$

$\dim(\operatorname{im} A) = \operatorname{rank}(A) = 3$

$\dim(\ker A) = 3 - 3 = 0$

b. (5 *pts*) $A = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$

Columns are colinear.

$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) \Rightarrow \operatorname{rank}(A) = 1$

$\dim(\ker A) = 3 - 1 = 2$

c. (5 *pts*) $\cancel{A = \begin{bmatrix}1 & 4 & 5 & 3\\ 0 & 2 & 7 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}}$ $A = \begin{bmatrix}1 & 4 & 5 & 3\\ 0 & 2 & 7 & 4\end{bmatrix}$

$\operatorname{im}(A) = \operatorname{span}$ (columns of A).

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}4\\2\end{bmatrix}$ are linearly indep $\Rightarrow \overbrace{\text{they span } \mathbb{R}^2}^{\text{form basis of } \mathbb{R}^2}$.

$\Rightarrow$ other col's are redundant $\Rightarrow$ $\dim(\operatorname{im} A) = 2$

$\dim(\ker A) = 4 - 2 = 2$

4 ~~5~~. (*23 points*) Determine whether each of the following transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear. If so, find its matrix.

a. (5 *pts*) $T(x, y) = (x + 1, y - 1, x + y)$,

Not linear. $T(0,0) = (1, -1, 0)$

$T(k(0,0)) = (1, -1, 0) \neq k(1, -1, 0)$

b. (8 *pts*) $T(x, y) = (x + 4y, 2x - 5y, 3x + 6y)$.

Linear:

$T(\vec{e}_1) = T(1,0) = (1, 2, 3)$

$T(\vec{e}_2) = T(0,1) = (4, -5, 6)$

Matrix : $\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{bmatrix}$

5 6. (16 points) Determine whether each of the following linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible. If so, find its inverse. You may describe the inverse either as a matrix or in words (e.g. "the rotation by $\pi/2$ counter-clockwise").

a. (8 pts) Orthogonal projection onto the line $y = 2x$ followed by rotation by $\pi/2$ counter-clockwise.

$$\underbrace{\text{projection}}_{P} \qquad \underbrace{\text{rotation}}_{R}$$

$$\ker(P) \neq \{\vec{0}\} \Rightarrow \ker(RP) \supset \ker(P) \neq \{0\}.$$

$\Rightarrow RP$ is not invertible.

b. (8 pts) Rotation by $\pi/4$ counter-clockwise followed by reflection about the x-axis.

$$\underbrace{\text{Rotation}}_{R} \qquad \underbrace{\text{reflection}}_{L}$$

$$(LR)^{-1} = R^{-1}L^{-1} \quad \text{invertible.}$$

The inverse is reflection about x-axis (L^{-1}) followed by a rotation by $\pi/4$ clock-wise (R^{-1}).

7. (23 points) Determine for which k is the matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ invertible, and find the inverse for those k.

$$\left[\begin{array}{cc|cc} 1 & k & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-k\cdot \mathrm{II}} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -k \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Hence A is invertible for all k, and

$$A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$

6. (23 points)

a. (10pts) Give an example of a matrix whose kernel is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

The equation can be written as $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

Hence: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. is a required matrix.

b. (10pts) Give an example of a matrix whose image is the plane $x + 2y + 3z = 0$ in $\mathbb{R}^3$.

$\text{im}(A) = \text{span}$ (columns of A).

We ~~thus~~ need a basis of the plane $x + 2y + 3z = 0$.

Choosing $y = 0$ and then $z = 0$, we see that the vectors

$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ belong to the plane and are linearly indep. $\Rightarrow$ form a basis of the plane.

Hence

$A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$ is a required matrix.

7 ~~8~~. (~~10~~ *points*) Let V be the linear space spanned by the two functions $\cos(t)$ and $\sin(t)$. Find the matrix of the transformation

$$T(f) = f(\pi/2 - t)$$

with respect to the basis $\cos(t)$ and $\sin(t)$.

$$\{ \underset{f_1}{\cos(t)}, \underset{f_2}{\sin(t)} \} = \mathcal{B}$$

$$T_{\mathcal{B}} = \begin{bmatrix} | & | \\ [T(f_1)]_{\mathcal{B}} & [T(f_2)]_{\mathcal{B}} \\ | & | \end{bmatrix}.$$

$$T(f_1) = T(\cos t) = \cos\left(\frac{\pi}{2} - t\right) = \sin t.$$

$$T(f_2) = T(\sin t) = \sin\left(\frac{\pi}{2} - t\right) = \cos t.$$

$$[\sin t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [\cos t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\boxed{T_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.}$$

8. (10 *points*) Find a 2×2 matrix A such that

$$A \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} = \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{w}_1} \quad \text{and} \quad A \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_2} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{w}_2}.$$

We will express A as $A = CB^{-1}$ where

$$\begin{tikzcd} \vec{v}_i \arrow[rr] & & \vec{w}_i \\ & \vec{e}_i \arrow[ul, "B"] \arrow[ur, "C"'] & \end{tikzcd}$$

B and C are coordinate transformations with respect to the bases $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{w}_1, \vec{w}_2\}$ respectively.

Thus $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$.

$B^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow A = CB^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -5 \end{bmatrix}.$