Criteria of Integrability

(bock: e.g. to prove that $f$ continuous on $(a,b)$ is integrable)

THM (Sequential Criterion)

$f \in \mathcal{R}[a,b] \iff \exists$ seq. of partitions $(P_n)$ s.t. $\|P_n\| \to 0, S(f, P_n)$ converges.

Moreover, in this case $\lim S(f, P_n) = \int_a^b f \quad \forall P_n, \|P_n\| \to 0.$

(⇒)

Assume (1) $f$ integrable; (2) $\|P_n\| \to 0.$

wts: $\lim S(f, P_n) = L.$

(∀ε>0 =)

(1) $\exists \delta > 0$: a partition $P$, $\|P\| < \delta$, satisfies $|S(f, P) - L| < \varepsilon.$

(2) $\forall \delta > 0 \exists N$: $\forall n > N$ we have $\|P_n\| < \delta.$

Choose $\delta$ from (1), put it into (2).

⇒ (∀N) $\forall n > N$ we have $\|P_n\| < \delta$ \implies $|S(f, P_n) - L| < \varepsilon.$

⇒ $S(f, P_n) \to L.$

QED

(⇐)

First note that

Assume the assumption in RHS holds.

⇒ Note that $S(f, P_n)$ converges to the same limit $V(P_n)$, i.e.

$\exists L: \forall (P_n) \text{ s.t. } \|P_n\| \to 0 \text{ we have } S(f, P_n) \to L.$ (⋆)

(Otherwise $\exists (P_n)(Q_n): \|P_n\| \to 0, \|Q_n\| \to 0, S(f, P_n) \to L_1, S(f, Q_n) \to L_2, L_1 \neq L_2$

Then the alternating seq. $R_n := (P_1, Q_1, P_2, Q_2, P_3, Q_3, \ldots)$ satisfies $\|R_n\| \to 0$

but $S(f, R_n)$ diverges)

⇒ Fix this $L$. N f not integrable ⇒ $\exists \varepsilon > 0 \forall \delta > 0 \exists P_n \text{ s.t. } \|P_n\| < \delta \text{ but } |S(f, P_n) - L| > \varepsilon.$

Apply this for $\delta_n := \frac{1}{n}$. ⇒ $\exists (P_n): \|P_n\| < \frac{1}{n}$ but $|S(f, P_n) - L| > \varepsilon.$

⇒ $\|P_n\| \to 0$ but $S(f, P_n) \not\to L$.

This contradicts (⋆). QED
Remark: If $f \in \mathbb{R}(a,b)$, we can approximate it using the uniform partition $x_1, x_2, \ldots, x_n$ and with tags $t_i$, e.g., at $x_i$. In particular, for $[a,b] = [0,1]$:

$x_i = \frac{i}{n}$, $t_i = \frac{i-\frac{1}{2}}{n}$.

\[
\frac{1}{n} \sum_{i=1}^{n} f \left( \frac{i}{n} \right) \to \int_{0}^{1} f(x) \, dx
\]

Discrete average $\to$ continuous average

Example:

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 = \frac{1}{3}
\]

(n \to \infty)

\[
\frac{1}{n} \frac{1}{2} \sum_{i=1}^{n} \frac{1}{n^2} i^2 \to \frac{1}{3}
\]

\[
\frac{1}{n^2} \sum_{i=1}^{n} i^2 \to \frac{1}{3}
\]

\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} + o(n^3)
\]

(Assumed, exact value $= \frac{n(n+1)(2n+1)}{6}$)
\[ \text{Def (Upper/lower Riemann sums)} \quad \text{given a part} \ P \]

\[
M_i := \sup \{ f(x) : x \in [x_i, x_{i+1}] \} \\
m_i := \inf \{ f(x) : x \in [x_i, x_{i+1}] \} \\
S^+(f, P) := \sum_{i=1}^n M_i (x_i - x_{i-1}) \\
S_+(f, P) := \sum_{i=1}^n m_i (x_i - x_{i-1})
\]

**Ex**

\[ \forall P, \quad S_+(f, P) \leq S(f, P) \leq S^+(f, P) \]

\[ \forall P \text{ untagged } \exists \text{ tags } \theta : \quad S^+(f, \theta) \leq S(f, P) \leq S^+(f, P) \]

\[ \forall \text{ tags } \theta : \quad S_+(f, \theta) \leq S(f, P) \leq S_+(f, P) + \varepsilon \]

**Thm (Darboux criterion)** \[ f \in R[a, b] \iff \forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \forall \text{ untagged } P, \ | ||P|| < \delta \implies |S^+(f, P) - S_+(f, P)| < \varepsilon \]

Moreover, in this case, \[ S^+(f, P) + \varepsilon \leq \int_a^b f \leq S_+(f, P) + \varepsilon \]

**Thm (Darboux criterion)** TFAE:

(i) \[ f \in R[a, b] \]

(ii) \[ \forall \varepsilon > 0 \exists \delta > 0 : \forall \text{ partition } P \text{ satisfying } ||P|| < \delta \]

\[ \text{we have } S^+(f, P) - S_+(f, P) < \varepsilon \]

(iii) \[ \forall \varepsilon > 0 \exists \text{ partition } P \text{ satisfying } S^+(f, P) - S_+(f, P) < \varepsilon \]

Moreover, if (i)-(iii) hold, we have

\[ S^+(f, P) - \varepsilon \leq \int_a^b f \leq S_+(f, P) + \varepsilon \]
Proof (i) $\Rightarrow$ (ii): Assume $f$ is integrable. 
\[ \int_a^b f = L. \]

$\Rightarrow \forall \varepsilon > 0 \exists \delta > 0 : \forall P, \|P\| < \delta$ satisfies \[ |L(f, P) - L| < \varepsilon/4. \]

By Exercise (prev. p.), \exists tags \: |L(f, P) - S^*(f, P)| < \varepsilon/4.

\[ \delta \Rightarrow |L^*(f, P) - L| < \varepsilon/2. \]

Similarly, \[ |L_S(f, P) - L| < \varepsilon/2. \]

\[ \Rightarrow |S^*(f, P) - S_S(f, P)| < \varepsilon. \quad \checkmark. \]

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Define \[ L^* := \inf \{ S^*_n(f, P) : \text{partitions } P \}, \quad L^*_\sup := \sup \{ S_n(f, P) : \text{partitions } P \}. \]

Choose (let $\varepsilon > 0$). Choose $P$ as in (iii), \[ S^*(f, P) - S_S(f, P) < \varepsilon. \]

Since \[ L_S < L < L^* \]

\[ \Rightarrow 0 \leq L^* - L \leq \varepsilon. \]

This holds $\forall \varepsilon \Rightarrow L^* = L = L$. ("Darboux integral")

Remains to show that $f$ is integrable, $\int_a^b f = L$.

Note: $\forall \varepsilon$, both $S(f, P)$ and $L$ are $\in [S_n(f, P), S^*(f, P)]$.

Apply (*) \[ \Rightarrow \int f S(f, P) - \gamma S^*(f, P) = \int f S^*(f, P) - S_S(f, P) < \varepsilon. \quad \forall \|P\| < \delta. \]

QED

Ex. Dirichlet function is not integrable.