Criteria of Integrebility

(Goal: e.g. to prove that $\forall$ continuous f on $[a,b]$ is integrable)

THM (Sequential Criterion)

$f \in R[a,b] \iff \forall$ seq. of partitions (P_n) s.t. $\|P_n\| \to 0$, $S(f,P_n)$ converges.

Moreover, in this case $\lim S(f,P_n) = \int_a^b f$. $\forall P_n$, $\|P_n\| \to 0$.

($\Rightarrow$) Assume (1) $\int_a^b f = L$ ~~is integrable~~; (2) $\|P_n\| \to 0$.

WTS: $S(f,P_n) \to L$ ~~converges~~.

- Let $\varepsilon > 0$ $\Rightarrow$
  (1) $\exists \delta > 0$: $\forall$ partition P, $\|P\| < \delta$, satisfies $|S(f,P) - L| < \varepsilon$.
  (2) $\forall \delta > 0$ $\exists N$: $\forall n > N$ we have $\|P_n\| < \delta$.
- Choose δ from (1), put it into (2).
  $\xRightarrow{(2)}$ $\exists N$. $\forall n > N$ we have $\|P_n\| < \delta$ $\xRightarrow{(1)}$ $|S(f,P_n) - L| < \varepsilon$.

  $\Rightarrow S(f,P_n) \to L$.

QED

($\Leftarrow$) ~~First note that~~
Assume the assumption in RHS holds.

- Note that $S(f,P_n)$ converges to the same limit $\forall (P_n)$, i.e.

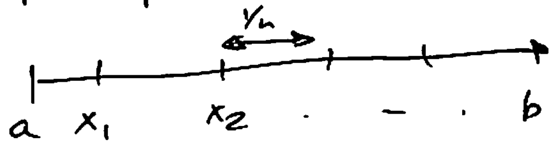
  $\exists L$: $\forall (P_n)$ s.t. $\|P_n\| \to 0$ we have $S(f,P_n) \to L$. (*)

  (Otherwise $\exists (P_n)(Q_n)$: $\|P_n\| \to 0$, $\|Q_n\| \to 0$, $S(f,P_n) \to L_1$, $S(f,Q_n) \to L_2$, $L_1 \neq L_2$.
  Then the alternating seq. $R_n := (P_1, Q_1, P_2, Q_2, P_3, Q_3) \ldots$ satisfies $\|R_n\| \to 0$
  but $S(f,R_n)$ diverges).

- Fix this L.
  If f not integrable $\Rightarrow \exists \varepsilon > 0$ $\forall \delta > 0$ $\exists P$ s.t. $\|P\| < \delta$ but $|S(f,P) - L| > \varepsilon$.
  Apply this for $\delta_n := \frac{1}{n}$. $\Rightarrow \exists (P_n)$: $\|P_n\| < \frac{1}{n}$ but $|S(f,P_n) - L| > \varepsilon$.
  $\Rightarrow \|P_n\| \to 0$ but $S(f,P_n) \not\to L$.

  This contradicts (*). QED

Remark If $f \in R[a,b]$, we can approximate it using the uniform partition



and with tags t_i e.g. at x_i.

In particular, for $[a,b] = [0,1]$: $\quad x_i = \frac{i}{n}, \quad t_i = \frac{i}{n}$.

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \longrightarrow \int_0^1 f}$$

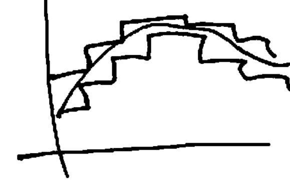
Discrete average $\longrightarrow$ continuous average

Ex $\quad \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \longrightarrow \int_0^1 x^2\,dx = \frac{1}{3}. \qquad (n \to \infty)$

$$\boxed{\frac{1}{n^3}\sum_{i=1}^{n} i^2 \longrightarrow \frac{1}{3}.} \qquad \Rightarrow \quad \sum_{i=1}^{n} i^2 = \frac{n^3}{3} + o(n^3).$$

(~~Actual~~, Exact value $= \frac{n(n+1)(2n+1)}{6}$)

Def (Upper/lower Riemann sums) Given a part. P (untagged)



$$M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$S^*(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$
$$S_*(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

Ex

- $\forall P$, $\quad S_*(f,P) \le S(f,P) \le S^*(f,P)$.
- $\forall P$ untagged $\exists$ tags : $\quad S^*(f,P) - \varepsilon \le S(f,P) \le S^*(f,P)$

  $\exists$ tags : $\quad S_*(f,P) \le S(f,P) \le S_*(f,P) + \varepsilon$ . .

~~THM (Darboux Criterion)~~ $\cancel{f \in R[a,b] \iff \forall \varepsilon > 0\ \exists \delta > 0}$

~~$\forall$ untagged P, $\|P\| < \delta$ satisfies $|S^*(f,P) - S_*(f,P)| < \varepsilon$.~~

~~Moreover In this case,~~ $\cancel{S^*(f,P) - \varepsilon \le \int_a^b f \le S_*(f,P) + \varepsilon}$.

THM (Darboux Criterion) TFAE:

(i) $f \in R[a,b]$

(ii) $\forall \varepsilon > 0\ \exists \delta > 0$: $\forall$ partition P satisfying $\|P\| < \delta$ we have $S^*(f,P) - S_*(f,P) < \varepsilon$

(iii) $\forall \varepsilon > 0\ \exists$ partition P satisfying $S^*(f,P) - S_*(f,P) < \varepsilon$.

Moreover, if (i)-(iii) hold, we have
$$S^*(f,P) - \varepsilon \le \int_a^b f \le S_*(f,P) + \varepsilon$$

Proof (i) $\Rightarrow$ (ii): Assume ~~f is integrable.~~ $\int_a^b f = L$.

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0$: $\forall P$, $\|P\| < \delta$ satisfies $|S(f,P) - L| < \varepsilon/4$

By Exercise (prev. p.), $\exists$ tags: $|S(f,P) - S^*(f,P)| < \varepsilon/4$.

$\Delta \Rightarrow |S^*(f,P) - L| < \varepsilon/2$.

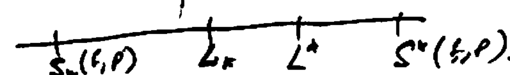
Similarly, $|S_*(f,P) - L| < \varepsilon/2$

$\Delta \Rightarrow |S^*(f,P) - S_*(f,P)| < \varepsilon$. ✓.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Define

$$L^* := \inf\{S^*(f,P) : \text{partitions } P\}, \quad L_* := \sup\{S_*(f,P) : \text{partitions } P\}$$



~~$\forall \varepsilon > 0$, choose~~

Let $\varepsilon > 0$. Choose P as in (iii). $\Rightarrow$ $S^*(f,P) - S_*(f,P) < \varepsilon$

Since $[L_*, L^*] \subset [S_*(f,P), S^*(f,P)]$ $\Rightarrow$

$$\Rightarrow 0 \le L^* - L_* < \varepsilon$$

This holds $\forall \varepsilon$ $\Rightarrow$ $L^* = L_* =: L$. ("Darboux integral")

Remains to show that f is integrable, $\int_a^b f = L$.

Note: $\forall P$, both $S(f,P)$ and L are $\in [S_*(f,P), S^*(f,P)]$.

Apply (*) $\Rightarrow$ $|S(f,P) - L| \le |S^*(f,P) - S_*(f,P)| < \varepsilon$. $\forall \|P\| < \delta$.

QED

Ex Dirichlet function is NOT integrable.