Math 35 Winter 2007

Real Analysis

Final Exam

Wednesday, March 7

Your name (please print): Text ____

Instructions: This is an open book, open notes, take home exam. You can also use any books you like. If you have any questions please contact me in person or via internet. **Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise in a given problem.

The exam is to be submitted by 11:59 PM on Wednesday March 14. **It is strongly preferred that you submit the exam in person.** However if you submit it at the time when my office is locked, then please write the time you finished working on the exam and slide your exam under the door of my office 304 Kemeny Hall.

The exam consists of 11 problems. Your total exam score is the sum of your scores for the 10 problems best solved. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /15

2. ________ /15

3. ________ /15

4. ________ /15

5. ________ /15

6. ________ /15

7. ________ /15

8. ________ /15

9. ________ /15

10. ________ /15

11. ________ /15

Total: ________ /150

(1) Let I be an interval and $f : I \to \mathbb{R}$ be a function such that

$$|f(y) - f(x)| \leq (\sin 1)|x - y|^{\frac{1}{2}},$$

for all $x, y \in I$. Prove that f is uniformly continuous on I.

(2) Prove that there is a point c in the interval $[0, \pi]$ such that the line tangent to the graph of $f(x) = \sin x + x^3 - \pi^2 x$ at $(c, f(c))$ has zero slope.

(3) Find the limit

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2 + \sin x}.$$

Explain your solution thoroughly.

(4) For the nonempty set $X \subset [1, 3]$ put $X^4 = \{x^4 | x \in X\}$. Let $s = \inf X$ and $t = \inf X^4$. Prove that $t = s^4$.

(5) Let $f : [0,1] \to \mathbb{R}$ be a function that is equal to 3 at all the points of $[0,1]$ except a point c where the value is some other finite number M. Use the definition of the Riemann integral through tagged partitions to show that f is Riemann integrable on $[0,1]$ and find the value of $\int_0^1 f$.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Is f continuous at $c = 0$? Prove your answer. (Hint: be careful.)

(7) For $k \in \mathbb{N}$ define $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(x) = 1 + \dfrac{\sin(e^{kx})}{k^3}$. Prove that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that the sequence of functions $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f on $\mathbb{R}$. Prove that the sequence of functions $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f on $\mathbb{R}$.

(8) Prove that the series $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{5k+1}$ converges. Put $S = \sum_{k=1}^{\infty} \frac{\cos(k\pi)}{5k+1}$ and for $n \in \mathbb{N}$ put $s_n = \sum_{k=1}^{n} \frac{\cos(k\pi)}{5k+1}$ to be the n-th partial sum. Find $n \in \mathbb{N}$ such that you are guaranteed that $|S - s_n| < 0.1$. Explain why your n satisfies the above condition. (Hint: you might want to write down the first few terms of the series explicitly to see what is going on.)

(9) Prove that the series $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^{1.1}}$ converges uniformly on $[0,2]$. (Hint: think before you try to apply something that does not apply.)

(10) Let $f : [0,1] \to \mathbb{R}$ be an increasing function (that is not continuous). Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in $[0,1]$. Prove that there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, such that the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges.

(11) Let $f_{\text{inv}} : \mathbb{R} \to \mathbb{R}$ be the inverse function of $f(x) = x^7 + x^5 + 3,\ f : \mathbb{R} \to \mathbb{R}$. Put $F(x) = \displaystyle\int_x^{-17} f_{\text{inv}}(t)dt$. Find the numerical value of $F'(5)$.