FINAL EXAM, ADVANCED CALCULUS I, FALL 2009

Name:

In all questions, explain your answer. In this test you can use without proofs theorems that were proved in the book, just give a reference.

1.(15 pts) Which two of the following inequalities have the property that a number x satisfies both of them if and only if $|x-3|<1$?

A: $|x-5|<2$

B: $|x-4|<2$

C: $|x+2|<3$

D: $|x-1|<3$

2. (15 pts) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Which one of the following statements implies that for every $\varepsilon > 0$ there exist $m, n \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$?

A: $\lim_{n\to\infty} x_n = -\infty$

B: the sequence (x_n) is bounded

C: for every $n \in \mathbb{N}$, $|x_{n+1} - x_n| < 0.01$

D: (x_n) is a decreasing sequence

3. (15 pts) Let $a \in \mathbb{R}$, and f, g are functions on $(a-1, a+1)$. Which two of the following statements, considered together, imply that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and are equal.

A: there exist $\varepsilon, \delta > 0$ such that if $|x-a| < \delta$ then $|f(x)-g(x)| < \varepsilon$

B: f is continuous at a

C: for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x)-g(x)| < \varepsilon$

D: there exists a sequence (x_n) such that $\lim_{n\to\infty} x_n = a$ and the limits $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} g(x_n)$ exist and are equal

4. (15 pts) Which two of the following statements, considered together, imply that $f(x) \geq g(x)$ for every $x \in [a, b]$.

A: f, g are differentiable on (a, b) and $f^{'}(x) \geq g^{'}(x)$ for every $x \in (a, b)$

B: there exists $c \in (a, b)$ such that $\lim_{x \to c}(f(x) - g(x)) \geq 0$

C: for every interval $(c, d) \subset [a, b]$ there exists $x \in (c, d)$ such that $f(x) \geq g(x)$

D: f, g are continuous on $[a, b]$.

5. (15 pts) A function f is defined on $[0,1]$ and has exactly two roots on $[0,1]$. Which of the following are possible?

A: f is increasing on $[0,1]$

B: there exists $c \in (0,1)$ such that f is increasing on each of the intervals $(0,c)$ and $(c,1)$

C: f is differentiable on $(0,1)$ and

$$f(0) = f(1/3) = f(2/3) = f(1)$$

D: f is continuous on $[0,1]$ and

$$f(0) = -1, f(1/3) = 1, f(2/3) = -1, f(1) = 1$$

6. (15 pts) Which one of the following sets is compact in $\mathbb{R}$?

A: (0,1]

B: $\{\frac{1}{n} : \; n \in N\}$

C: $[1, \infty)$

D: $\{0\} \cup \{-\frac{1}{n^2} : n \in \mathbb{N}\}$

7. (20 pts) Prove by induction (starting with $n = 2$) that for every $n \geq 2$

$$5^n \geq (2n + 3)3^{n-1} + 2^n$$

8. (20 pts) Prove using the definition of limit (find $N(\varepsilon)$ for every $\varepsilon > 0$) that

$$\lim_{n\to\infty} \frac{2n+1}{3n-1} = \frac{2}{3}$$

9. (20 pts) A function f is defined by $f(x) = x\sin(\frac{1}{x}),\ x \neq 0$ and $f(0) = 0$. Is f continuous at the origin? Is f differentiable at the origin?

10. (20 pts) A function f is continuous on $[0, 2]$ and differentiable on on $(0, 1)$ and on $(1, 2)$. Suppose that $f^{'}(x) \leq 1$ for every $x \in (0, 1)$, and $f^{'}(x) \leq 2$ for every $x \in (1, 2)$. Prove that

$$f(2) - f(0) \leq 3.$$

11. (20 pts) Suppose that f is an integrable function on $[0, \pi/2]$, and for every interval $(c, d) \subset [0, \pi/2]$ there exists a point $x \in (c, d)$ such that $f(x) \leq \sin x$. Prove that

$$\int_0^{\pi/2} f \leq 1.$$

12. (20 pts) Let G be an open in $\mathbb{R}$ and suppose that for every $n \in \mathbb{N}$ the interval $(0, \frac{1}{n})$ contains a point that does not belong to G. Prove that $0 \notin G$.

Extra Credit (10 pts). Suppose that (x_n) is a sequence of non-negative numbers such that $x_1 = 1$ and for every $n \in \mathbb{N}$

$$x_{n+2} < \frac{x_n}{2} + \frac{x_{n+1}}{4}$$

Prove that the series

$$\sum_{n=1}^{\infty} \frac{5^n x_n}{4^n}$$

converges.