Ex.: Similarly to Seq. Criterion, deduce:

THM (Darboux Sequential Criterion) TFAE:

(i) $f \in R[a,b]$

(ii) $\forall$ seq. of partitions (P_n) s.t. $\|P_n\| \to 0$, $S^*(f,P_n) - S_*(f,P_n) \to 0$

(iii) $\exists$ " ——————————————————————— "

Moreover, in this case $\lim S^*(f,P_n) = \lim S_*(f,P_n) = \int_a^b f$.

Integrability of Continuous & Monotone Functions.

THM (Continuous) $\forall$ continuous function $f : [a,b] \to \mathbb{R}$ is integrable

Will use Darboux criterion.

- f continuous on $[a,b]$ $\Rightarrow$ uniformly continuous.

$\forall \varepsilon > 0 \ \exists \delta > 0$:

$$\forall x, y \in [a,b],\ |x-y| < \delta \text{ satisfy } |f(x) - f(y)| < \frac{\varepsilon}{b-a}. \qquad (*)$$

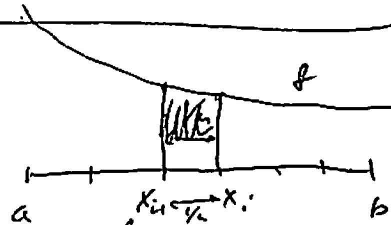
- To check Darboux, let P be a partition of $[a,b]$ with $\|P\| < \delta$.

Then, by $(*)$, $M_i - m_i < \frac{\varepsilon}{b-a}$ (here we used that M_i, m_i are attained $\forall i$)

$$\Rightarrow S^*(f,P) - S_*(f,P) \le \sum_{i=1}^n \underbrace{(M_i - m_i)}_{< \frac{\varepsilon}{b-a}} \underbrace{(x_i - x_{i-1})}_{> 0} \le \frac{\varepsilon}{b-a} \sum_{i=1}^n x_i - x_{i-1}$$

$$= \frac{\varepsilon}{b-a}(\underset{b}{x_n} - \underset{a}{x_0}) = \varepsilon. \qquad \text{QED}$$

$\underline{\text{THM}}$ (Monotone) $\forall$ monotone function $f : [a,b] \to \mathbb{R}$ is integrable



Will use Darboux sequential criterion.

Choose $P_n :=$ uniform partition of $[a,b]$, i.e. $x_i - x_{i-1} = \frac{b-a}{n} \ \forall i$

WLOG f is decreasing.

$$S^*(f, P_n) = \sum_{i=1}^{n} \underbrace{M_i}_{= f(x_{i-1})} \underbrace{(x_i - x_{i-1})}_{= \frac{b-a}{n}} = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$$

$$S_*(f, P_n) = \sum_{i=1}^{n} \underbrace{m_i}_{= f(x_i)} \underbrace{(x_i - x_{i-1})}_{= \frac{b-a}{n}} = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

$$\Rightarrow S^*(f, P_n) - S_*(f, P_n) = \frac{b-a}{n} \left[f(x_0) - f(x_n) \right]$$

$$= (b-a) \left[f(a) - f(b) \right] \cdot \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

QED

Restriction & combination of functions

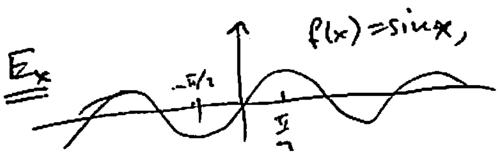
$\underline{\text{Def}}$: $f : D \to \mathbb{R}$, $I \subset D$.

$f|_I : I \to \mathbb{R}$ is defined by $f|_I(x) = f(x)$

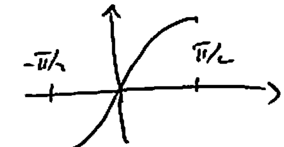
Related, but different from $f \cdot \mathbb{1}_I = \begin{cases} f(x), & x \in I \\ 0, & x \notin I. \end{cases}$

$f \cdot \mathbb{1}_I : D \to \mathbb{R}$

$\underline{\text{Ex}}$ $f(x) = \sin x$, $f : \mathbb{R} \to \mathbb{R}$.



$f|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$:



one-to-one.

$f \cdot \mathbb{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]} =$

Ex $f|_{[c,d]}$ is integrable $\iff$ $f\cdot \mathbb{1}_{[c,d]}$ is integrable

We say "f is integrable on $[c,d]$" if $f|_{[c,d]} \in R[c,d]$.

THM (Restriction)

If f is integrable on $[a,b]$ then f is integrable on $[c,d]$ $\forall\ [c,d] \subset [a,b]$

$\ulcorner$ Will use Darboux Crit.

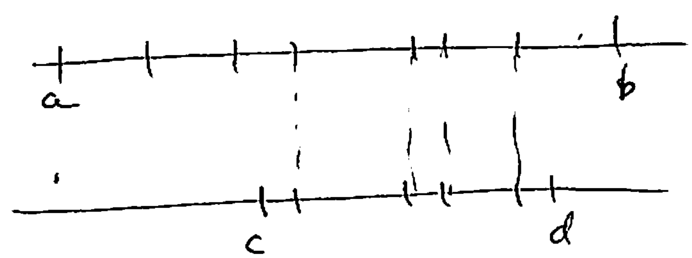
• $\forall \varepsilon > 0\ \exists \delta > 0:$ $\forall$ partition $P = \{x_i\}$ of $[a,b]$, $\|P\| < \delta$, satisfies

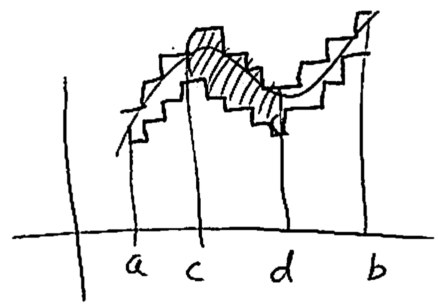
$$S^*(f,P) - S_*(f,P) < \varepsilon.$$

~~WLOG, $c,d \in P$ (otherwise add them; $\|P\|$ may only decrease).~~

Consider the restriction of P onto $[c,d]$:

$$P' := \left(\{x_i\} \cap [c,d]\right) \cup \{c,d\}$$





Then

$$\underbrace{S^*(f,P') - S_*(f,P')}_{} \le S^*(f,P) - S_*(f,P) < \varepsilon.$$

this is a sum of some terms included here. $\left(\sum \underbrace{(M_i - m_i)(x_i - x_{i-1})}_{\geq 0} \right)$

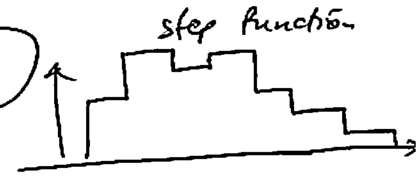
QED.

THM (Combination)

If f is integrable on $[a,c]$ & on $[c,b]$ then f is int. on $[a,b]$; $\int_a^b f = \int_a^c f + \int_c^b f$.

$\ulcorner$ $f = f \cdot \mathbb{1}_{[a,c]} + f \cdot \mathbb{1}_{[c,b]}$; finish by the sum thm. $\lrcorner$

Cor $\forall$ step function (with finitely many steps) is integrable



$\ulcorner$ A constant function is integrable

$\Rightarrow$ By restriction thm, $\forall$ step is integrable

$\Rightarrow$ By combination thm, the step function is integrable.

$\lrcorner$

THM (Approximation of integrable functions by step functions)

Let f be integrable on $[a,b]$.

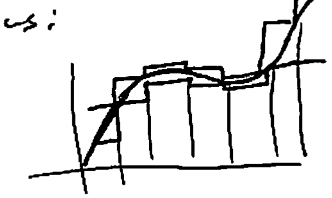
Then $\forall \varepsilon > 0$ $\exists$ step functions g, h on $[a,b]$ s.t.

$$g \le f \le h \quad \text{and} \quad \int_a^b (h-g) < \varepsilon.$$

- In particular, $\int h - \varepsilon \le \int f \le \int h$ and $\int g \le \int f \le \int g + \varepsilon$.

  (the integral of f is approximated by the integrals of g, h.)

$\ulcorner$ Build g, h from $S^*(f,P)$ & $S_*(f,P)$ as follows:



$h(x) := M_i$ when $x \in [x_{i-1}, x_i)$

$g(x) := m_i$ when $x \in [x_{i-1}, x_i)$.

Then $g \le f \le h$.

By Cor, h and g are integrable on $[a,b]$

By combination Thm, $\int_a^b h = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f = \sum_{i=1}^n M_i (x_i - x_{i-1}) = S^*(f,P)$.

and, similarly, $\int_a^b g = S_*(f,P)$.

We finish by Darboux criterion.

$\lrcorner$