MIDTERM EXAM SOLUTIONS

(1) (a) $\inf S = -1$, $\sup S = 1$

(b) $\inf S = -\infty$, $\sup S = +\infty$

(c) $\inf S = -1$, $\sup S = 0$ (observe that $S = (-1, 0)$)

(2) (a) $\frac{5}{3}$

(b) $+\infty$

(c) NOT EXIST (observe that $(s_n) = (\underbrace{1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}}_{\text{this part is repeated infinitely many times}}, \underbrace{1, \ldots}_{})$)

(3) (a) Let P_n be the assertion "$s_n \leq 1$".

P_1 : $0 \leq 1$ is true.

Assume P_n is true. Then

$$s_{n+1} = \frac{1}{2}(s_n + 1) \underset{\text{by } P_n}{\leq} \frac{1}{2}(1+1) = 1,$$

hence P_n is true.

Therefore, by the P.M.I, $s_n \leq 1$ for all $n \in \mathbb{N}$.

(b) $s_{n+1} = \underbrace{\frac{1}{2}(s_n + 1) \geq s_n}_{}$ for all $n \geq 1$?

This inequality is equivalent to

$1 \geq s_n$,

which holds for all n by part (a).

(c) (s_n) is <u>monotone</u> (by part (b)) and <u>bounded</u>: it is bounded above (by part (a)) and below because ~~[illegible]~~ $s_n \geq s_1 = 0$ for all $n \in \mathbb{N}$ (as (s_n) is nondecreasing.

Thus by one of the limit theorems, (s_n) converges.

Let $s = \lim s_n = \lim s_{n+1}$. We have

$$\lim s_{n+1} = \lim \frac{1}{2}(s_n+1) = \frac{1}{2}(\lim s_n + 1),$$

hence

$$s = \frac{1}{2}(s+1).$$

Solving this for s, we obtain

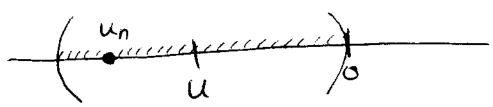
$$s = 1.$$

Answer: $\lim s_n = 1$.

(4) (a) 1st solution. Introduce the sequence $u_n = t_n - s_n$. Then part (a) reduces to proving that

$$u_n \geq 0 \text{ for all } n \quad \text{implies} \quad u = \lim u_n \geq 0. \qquad (*)$$

By contraposition, assume that $u < 0$.



By the definition of the limit with $\varepsilon = -u > 0$, there exists an N such that

$$|u_n - u| < -u \quad \text{whenever } n > N$$

which implies $\quad u_n - u < -u$

hence $\quad u_n < u - u = 0 \quad$ for $n > N$.

Contradiction. Hence $(*)$ holds, and part (a) is proved.

(b) No. For example, let $s_n = 0$, $t_n = \frac{1}{n}$ for all n. Then $s_n < t$ but $\lim s_n = \lim t_n = 0$.

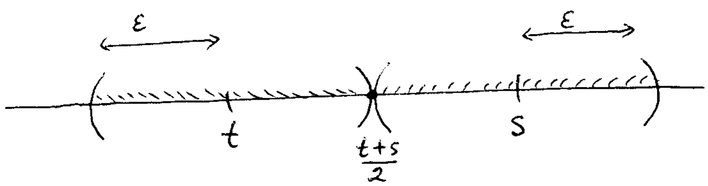
Alternative solution to 4a (a direct proof)

Let $s = \lim s_n$ and $t = \lim t_n$

$\forall \varepsilon > 0 \ \exists N$ such that $|s_n - s| < \varepsilon$ and $|t_n - t| < \varepsilon$ whenever $n > N$.

(this follows from the definition of the limit, where we take $N = \max(N_1, N_2)$ for N_1 corresponding to s and N_2 corresponding to t, as in the lectures).

By contraposition, assume that $s > t$. Take $\varepsilon = \frac{1}{2}(s-t) > 0$.



For $n > N$ we have

$$s - \varepsilon < s_n < s + \varepsilon, \qquad t - \varepsilon < t_n < t + \varepsilon$$

Hence

$$s_n > s - \tfrac{1}{2}(s-t) = \frac{t+s}{2} = t + \frac{(s-t)}{2} > t_n.$$

Contradiction. Thus $s \le t$, as required.

<u>Intuitively</u>, for $n > N$ the terms s_n must belong to the right interval in the picture above, while the terms t_n must belong to the left interval. All numbers in the right interval are strictly larger than those in the left interval. Hence $s_n > t_n$, which contradicts to the assumption