

Midterm Exam 1

SOLUTIONS

Math 451, Prof. Roman Vershynin
Fall 2011

Name: _____

Read the following information before starting the exam:

- No laptops or any communication devices are allowed on the exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

Problem	Points
1	
2	
3	
4	
5	
6 (bonus)	
Total	

1. (10 points) For each of the following sequences, compute $\inf\{s_n\}$, $\sup\{s_n\}$, $\liminf s_n$ and $\limsup s_n$. No justification is necessary; you may just write down the answers.

a. (5 pts) $s_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$ $s_n = \begin{cases} \frac{1}{n} + 1 & \text{for even } n, \\ -\frac{1}{n} & \text{for odd } n. \end{cases}$

$\inf\{s_n\} = -1$, $\liminf\{s_n\} = 0$

$\sup\{s_n\} = \frac{3}{2}$, $\limsup\{s_n\} = 1$

b. (5 pts) $s_n = -n[2 + (-1)^n]$ $s_n = \begin{cases} -3n & , n \text{ even} \\ -n & , n \text{ odd} \end{cases}$

$\inf\{s_n\} = -\infty$, $\liminf\{s_n\} = -\infty$

$\sup\{s_n\} = -1$, $\limsup\{s_n\} = -\infty$.

2. (20 points) Let (s_n) be a convergent sequence and let $\lim s_n = s$. Show that $(|s_n|)$ is a convergent sequence and $\lim |s_n| = |s|$. Use the definition of limit in your argument.

Let $\epsilon > 0$. Since $\lim s_n = s$, there exists N such that

$$|s_n - s| < \epsilon \quad \text{for all } n > N.$$

Then $||s_n| - |s|| \leq |s_n - s| < \epsilon$ for $n > N$ by triangle inequality.

Therefore $\lim |s_n| = |s|$. \square

3. (20 points) Consider nonempty bounded subsets A and B of positive real numbers, and define

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}.$$

(That is, a number z is in $A \cdot B$ if $z = ab$ for some $a \in A$ and $b \in B$.) Show that

$$\sup(A \cdot B) = \sup A \cdot \sup B. \quad (*)$$

(*) is equivalent to :

- $$\left\{ \begin{array}{l} \text{(i) } ab \leq \sup A \cdot \sup B \text{ for all } a \in A, b \in B. \\ \text{(ii) For each } M' < \sup A \cdot \sup B, \text{ there exist } a \in A, b \in B \text{ such that } ab > M'. \end{array} \right.$$

Now, (i) follows by def. of sup, since $a \leq \sup A$, $b \leq \sup B$.

To prove (ii), let $M' = \sup A \cdot \sup B - \epsilon'$ for some ϵ' .

For every $\epsilon > 0$, there exist $a \in A$, $b \in B$ such that $a \geq \sup A - \epsilon$,
 $b \geq \sup B - \epsilon$.

$$\begin{aligned} \Rightarrow ab &\geq (\sup A - \epsilon)(\sup B - \epsilon) \\ &= \sup A \cdot \sup B - (\sup A + \sup B) \cdot \epsilon + \epsilon^2. \end{aligned}$$

Note that $\sup A + \sup B > 0$, by assumption,

and choose $\epsilon = \frac{\epsilon'}{\sup A + \sup B}$. Then

$$ab \geq \sup A \cdot \sup B - \epsilon', \text{ as required}$$

QED.

4. (30 points) Consider the sequence (a_n) defined recursively as

$$a_1 = 1, \quad a_n = \sqrt{3a_{n-1} - 2} \quad \text{for } n = 2, 3, \dots$$

a. (10 pts) Show by induction that

$$1 \leq a_n \leq 2 \quad \text{for all } n = 1, 2, \dots$$

The assertion is trivially true for $n=1$, as $a_1 = 1$.

Assume $1 \leq a_{n-1} \leq 2$ for some $n \in \mathbb{N}$.

Then multiplying by 3, subtracting 2 from all sides yields

$$1 \leq 3a_{n-1} - 2 \leq 4.$$

Taking the square root gives

$$1 \leq \sqrt{3a_{n-1} - 2} \leq 2,$$

so $1 \leq a_n \leq 2$ is proved.

Q.E.D.

b. (10 pts) Show by induction that the sequence (a_n) is non-decreasing.

$$a_n \geq a_{n-1} \quad \text{is equivalent to} \quad \sqrt{3a_{n-1} - 2} \geq a_{n-1}, \quad (*)$$

which is further equivalent to $a_{n-1}^2 - 3a_{n-1} + 2 \leq 0$ (using that both sides of (*) are non-negative).

Since the inequality $x^2 - 3x + 2 \leq 0$ is equivalent to $1 \leq x \leq 2$.

Since $1 \leq a_n \leq 2$ by part (a), inequality (*) is satisfied, Q.E.D.

~~Q.E.D.~~

c. (10 pts) Deduce that the sequence (a_n) converges and find its limit.

Since (a_n) is monotone and bounded by (a) and (b), (a_n) converges.

Let $a = \lim a_n$. Taking the limit of both sides of

$$a_n^2 = 3a_{n-1} - 2$$

and using limit theorems, we conclude that

$$a^2 = 3a - 2.$$

Solving this yields $a = 1$ or $a = 2$

But in fact $a_n = 1$ for all n (by induction: $a_1 = 1$;

if $a_{n-1} = 1$ then $a_n = \sqrt{3 \cdot 1 - 2} = 1$).

So $\lim a_n = 1$.

5. (20 points) Compute the following limits. Justify all steps. If you apply a theorem on the limit of a sum, product, or ratio, you may put "by a limit theorem" without specifying its number. Other theorems need to be specified (the theorem's number in the book or the page number in class notes is OK).

$$\text{a. (10 pts) } \lim \frac{7n - \sin(\pi n^2) + 1}{3n + \cos(\pi n^2)} = \lim \frac{7 - \frac{\sin(\pi n^2)}{n} + 1/n}{3 + \frac{\cos(\pi n^2)}{n} + \frac{1}{n}}$$

~~Since~~ ~~lim~~ ~~sin~~ Since $(\sin(\pi n^2))$ and $(\cos(\pi n^2))$ are bounded sequences, ~~and~~ $\lim \frac{\sin(\pi n^2)}{n} = \lim \frac{\cos(\pi n^2)}{n} = 0$; also $\lim \frac{1}{n} = 0$.

$$\Rightarrow \text{answer} = \boxed{7/3}$$

$$\text{b. (10 pts) } \lim (\sqrt{n+\sqrt{n}} - \sqrt{n}) \neq \lim \frac{(\sqrt{n+\sqrt{n}} - \sqrt{n}) - 0}{(\sqrt{n+\sqrt{n}} - \sqrt{n})(\sqrt{n+\sqrt{n}} + \sqrt{n})}$$

$$= \lim \frac{(\sqrt{n+\sqrt{n}} - \sqrt{n})(\sqrt{n+\sqrt{n}} + \sqrt{n})}{\sqrt{n+\sqrt{n}} + \sqrt{n}} = \lim \frac{n+\sqrt{n} - n}{\sqrt{n+\sqrt{n}} + \sqrt{n}} =$$

$$= \lim \frac{1}{\underbrace{\sqrt{1 + \frac{1}{\sqrt{n}}}}_{\downarrow 1} + 1} = \lim \frac{1}{1+1} = \boxed{\frac{1}{2}} \quad (\text{by Example 5 p. 40})$$

6. (10 points) [Bonus problem, no partial credit] Let a and b be positive real numbers. Compute

$$\lim \sqrt[n]{a^n + b^n}.$$

Case 1: let $a \leq b$. Then

$$\lim \sqrt[n]{a^n + b^n} = \lim b \sqrt[n]{\underbrace{\left(\frac{a}{b}\right)^n + 1}_{\downarrow 0}} = b.$$

(Here we use that $1 \leq \sqrt[n]{\left(\frac{a}{b}\right)^n + 1} \leq \left(\frac{a}{b}\right)^n + 1$, so $\sqrt[n]{\left(\frac{a}{b}\right)^n + 1} \rightarrow 1$ by Squeeze Thm.)

Case 2: $a > b$. Analogously, $\lim \sqrt[n]{a^n + b^n} = a$.

Case 3: $a = b$. Then $\lim \sqrt[n]{a^n + b^n} = \lim \sqrt[n]{2a^n} = \sqrt[n]{2} a$
 $= a \cdot \lim \sqrt[n]{2} = a$ (by 9.7 Ross (d))

Therefore, in either case,

$$\boxed{\lim \sqrt[n]{a^n + b^n} = \max(a, b)}$$