MIDTERM EXAM 2 (SOLUTIONS)

(MATH 451, FALL 2011)

1

(a) Diverges by the Limit Comparison Test (p. 47 of notes).

Compare $a_n = \frac{1}{\sqrt[3]{n^2+1}}$ to $b_n = \frac{1}{\sqrt[3]{n^2}} = n^{-2/3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)^{1/3} = \lim_{n\to\infty}\left(\frac{1}{1+n^{-2}}\right)^{1/3} = 1 \text{ (by limit theorems).}$$

Since $\sum b_n = \sum n^{-2/3}$ diverges (Ex. (b) p. 47 of notes),

$\sum a_n$ diverges as well.

(b). Converges by the Root Test.

For $a_n = \frac{1}{(\log n)^n}$, $\quad a_n^{1/n} = \frac{1}{\log n} \to 0 \quad (n\to\infty)$.

Hence $\sum a_n$ converges.

(c) Diverges by Ratio Test.

For $a_n = \frac{3^n n!}{n^n}$, $\quad \frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = 3\left(\frac{n}{n+1}\right)^n$

$$= 3\left(1-\frac{1}{n+1}\right)^n \to \frac{3}{e} > 1. \text{ (see Ex. (b) p. 44)}$$

Hence $\sum a_n$ diverges.

(d) Converges by Alternating Series Theorem, since $\frac{1}{\log n} \geq 0$ for $n \geq 2$ and $\frac{1}{\log(n)}$ is a monotonically decreasing sequence.

(indeed, $\log(m) \geq \log(n) \iff m \geq n$ - by exponentiating, and since e^x increases in x as $(e^x)' = e^x > 0$).

(e) Diverges by the Limit Comparison Test.

Indeed, we compare $a_n = \frac{1}{n^{1+\frac{1}{n}}}$ to $b_n = \frac{1}{n}$.

$\frac{a_n}{b_n} = \frac{1}{n^{1/n}} \to 1$ as $n \to \infty$ (Prop. 9.7(c))

Since $\sum b_n = \sum \frac{1}{n}$ diverges, $\sum a_n$ diverges as well.

2

(a) . True by the Comparison Test (Thm 14.6):

$\sum a_n$ diverges and $\max(a_n, b_n) \geq a_n$ implies that $\sum \max(a_n, b_n)$ diverges.

(b) False. Let $a_n = (-1)^n + 1$ and $b_n = (-1)^{n+1} + 1$.

Then $\sum a_n$ and $\sum b_n$ diverge (since $a_n \not\to 0$, $b_n \not\to 0$: Divergence test) while $\sum \min(a_n, b_n) = \sum 0 = 0$ converges.

3

- $f(x) = \sin^3 x - k\cos x$ is a continuous function on $\mathbb{R}$, and

$$f(0) = -k, \qquad f(\pi/2) = 1.$$

Therefore, if $k > 0$ then by Intermediate Value Theorem, the equation $f(x)=0$ has a solution in $(0, \pi/2)$.

- Next, if $k=0$ then the equation becomes $\sin^3 x = 0$, which has solution $x=0$.
- Finally, if $k<0$ then $\sin^3 x - k\cos x > 0$ for all $x \in [0, \pi/2]$ $\Rightarrow$ no solution.

Summarizing:

ANSWER: $\boxed{k \geq 0}$.

4

(a) $\lim_{x\to 1} \frac{x^p-1}{x^q-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{p-1}+x^{p-2}+\ldots+1)}{\cancel{(x-1)}(x^{q-1}+x^{q-2}+\ldots+1)} \ominus$

The enumerator and denominator each are polynomials of degrees $p-1$ and $q-1$ respectively. These are continuous functions and the denominator $\neq 0$ at $x=1$. Hence by the limit of ratios,

$$\ominus \frac{\lim_{x\to 1} \overbrace{(x^{p-1}+\ldots+1)}^{p \text{ terms}}}{\lim_{x\to 1} \underbrace{(x^{q-1}+\ldots+1)}_{q \text{ terms}}} = \boxed{\frac{p}{q}}.$$

(b) Using that $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ (Hw9, Additional required problem #1c),

we have

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{1-\cos x} \cdot \frac{\sin x}{x} = \frac{1}{2}\cdot 1$$ (by limit of a product and by the classical limit).

$$= \boxed{\frac{1}{2}}$$

ANSWER: e^{2a}.

(c) W.L.O.G $a \neq 0$ (otherwise $\lim = 1$). Also, WLOG $a>0$ (otherwise consider $\lim \frac{1}{a_n} = \lim\left(\frac{x-a}{x+a}\right)^x$.

$$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty}\left(1+\frac{2a}{x-a}\right)^x = \lim_{y\to\infty}\left(1+\frac{2a}{y}\right)^{y+a}$$ (by change of var. $x = y-a$)

$$= \lim_{y\to\infty}\left(1+\frac{2a}{y}\right)^y \underbrace{\left(1+\frac{2a}{y}\right)^a}_{\downarrow\, 1}.$$

1 as $y\to\infty$ (as $\frac{2a}{y}\to 1$; the function $f(z)=z^a$ is continuous at 1).

$$\left(1+\frac{2a}{y}\right)^y = \left[\underbrace{\left(\left(1+\frac{2a}{y}\right)^{\frac{y}{2a}}\right)}_{\downarrow}\right]^{2a}$$

e by Ex.(c) p.65, as $\frac{2a}{y}\to\infty$

$\longrightarrow e^{2a}$ (as the function $f(z)=z^{2a}$ is continuous at $z=e$).

ANSWER: $\boxed{e^{2a}}$.

(d) $\lim_{x\to 0^+} (\sin x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} e^{\frac{\ln \sin x}{\ln x}}$

Consider separately

$$\frac{\ln \sin x}{\ln x} = \frac{\ln\left(\frac{\sin x}{x}\cdot x\right)}{\ln x} = \frac{\ln\left(\frac{\sin x}{x}\right) + \ln x}{\ln x} = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln x} + 1. \qquad (*)$$

Since $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, and $\ln(z)$ is continuous at 1,

$$\lim_{x\to 0^+} \ln\left(\frac{\sin x}{x}\right) = \ln(1) = 0 . \qquad \text{(by ~~comp~~ limit of composition)}$$

Since $\lim_{x\to 0^+} \ln x = -\infty$, $\quad \lim_{x\to 0^+} \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln x} = 0$. (Ratio limit)

$$\Rightarrow \lim \frac{\ln \sin x}{\ln x} = 0 + 1 = 1 \quad \text{by } (*)$$

Since e^z is continuous at 1,

$$\lim_{x\to 0^+} e^{\frac{\ln \sin x}{\ln x}} = e^1 = \boxed{e} \qquad \text{(By limit of composition).}$$

ANSWER: e.

5

(a) $f(x) = \frac{x-1}{x+1}$ is continuous on $[0,1]$ since $x+1 \neq 0$ on $[0,1]$ and $x-1$, $x+1$ are continuous (thm on the continuity of a ratio).

Hence f is uniformly continuous on $[0,1]$ (Thm 19.2).

(b) $f(x) = \ln x$ is NOT unif. continuous on $(0,1)$ by Lemma (p. 59 of notes).
Let $x_n = \frac{1}{n} \in (0,1)$, $n \in \mathbb{N}$. Then (x_n) is Cauchy. However,
$\ln x_n = -\ln n \to -\infty \Rightarrow (\ln x_n)$ is unbounded, hence not Cauchy (Lemma 10.10).
Thus $f(x) = \ln x$ is not unif. continuous on $(0,1)$.

(c) e^x is NOT unif. continuous on $(-\infty,\infty)$ by Lemma (p. 59 of notes)
Let $x_n = \ln n$, $y_n = \ln(n+1)$. Then:

$$|x_n - y_n| = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \underbrace{\tfrac{1}{n}}_{0}\right) \to \ln(1) = 0 \qquad \text{(by continuity of } \ln(z))$$

but

$$e^{x_n} - e^{y_n} = n - (n+1) = -1 \not\to 0 .$$

Hence e^x is not unif. continuous on $(-\infty,\infty)$. QED.

(6)

W.L.O.G. f is non-decreasing on $[0,1)$. Then

$$f(x) \le f(x_0) \le f(y) \qquad \text{whenever} \quad x \le x_0 \le y.$$

Then, by a comparison theorem (similar to Ex. 2 in HW9 Additional Problems),

$$\lim_{x \to x_0-} f(x) \le f(x_0) \le \lim_{y \to x_0+} f(y)$$

The limits in LHS and RHS both exist and equal $\lim_{x \to x_0} f(x)$ (which exists by assumption).

Hence, by squeeze theorem,

$$\lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f \text{ is continuous at } x_0. \qquad QED.$$

(7)

Let $x \in \mathbb{R}$ be arbitrary.
By the denseness of $\mathbb{Q}$ (4.7) and of $\mathbb{I}$ (Ex. 4.12) in $\mathbb{R}$, there exist sequences $x_n \in \mathbb{Q}$, $y_n \in \mathbb{I}$ such that

$$\lim x_n = \lim y_n = x.$$

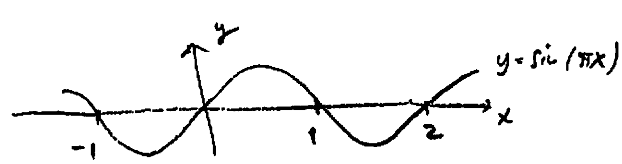
If f were continuous at x, we would have

$$\lim f(x_n) = \lim f(y_n) = f(x) \qquad (*)$$

However, $\lim f(x_n) = \lim 1 = 1$ while $\lim f(y_n) = \lim 0 = 0$ (by def. of f), so $(*)$ fails. This contradiction implies that f is discontinuous at x.

(8)

<u>Example</u>: $f(x) = \begin{cases} \sin(\pi x), & x \in \mathbb{Q} \\ 0, & x \in \mathbb{I} \end{cases}$:



(a) Discontinuous at each $x \notin \mathbb{Z}$.
Indeed, $\sin(\pi x) \ne 0$, so an argument similar to #7 $(x_n, y_n \to x)$ shows that f is discontinuous at x.

(b) Continuous at each $x \in \mathbb{Z}$.
Indeed, here $\sin(\pi x) = 0$. Let $x_n \to x$. Since $f(x_n) \in \{0, \sin(\pi x_n)\}$,

$$|f(x_n)| \le |\sin(\pi x_n)| \to |\sin(\pi x)| = 0 \qquad \text{(by continuity)}$$

it follows by squeeze thm that

$$f(x_n) \to 0 = f(x).$$

Hence f is continuous at $x \in \mathbb{Z}$. QED