Midterm Exam 2

Math 451, Prof. Roman Vershynin

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Read the following information before starting the exam:

- You may use textbooks and your own course notes, both from this course and from any other course you have taken. You may **not** use any electronic or communication devices, in particular internet and calculators. This should be strictly individual work; please do not consult with any other person.
- Your solutions may refer only to results from Ross textbook or from class notes. Please indicate the exact location of the result you are citing (e.g. Theorem 12.4 from Ross, or: Example (b) from class notes 10/24/2011). When you refer to limit theorems for sequences, series or functions, you may just say "by the limit of a sum" etc.
- If possible, please use plain white paper (as opposed to lined paper).
- Show all work. Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*25 points*) For each of the following series, determine whether it is convergent or divergent. Justify.

a. (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

b. (*5 pts*) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

c. (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

d. (*5 pts*) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$

e. (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

2. (*12 points*) Let $\sum a_n$ and $\sum b_n$ be divergent series with non-negative terms. For each of the following statements, determine whether it is true or false. Prove or give a counterexample.

a. (*6 pts*) The series $\sum \max(a_n, b_n)$ diverges.[1]

b. (*6 pts*) The series $\sum \min(a_n, b_n)$ diverges.

3. (*7 points*) Find all values of $k \in \mathbb{R}$ for which the equation

$$\sin^3 x - k\cos x = 0$$

has a solution in $[0, \pi/2]$. Justify.

[1] $\max(a, b)$ denotes the larger of the numbers a and b, and $\min(a, b)$ denotes the smaller of the numbers a and b.

4. (*24 points*) Compute the following limits. Justify all steps.

a. (*6 pts*) $\lim_{x \to 1} \frac{x^p - 1}{x^q - 1}$ where $p, q \in \mathbb{N}$.

b. (*6 pts*) $\lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$

c. (*6 pts*) $\lim_{x \to +\infty} \left(\frac{x + a}{x - a} \right)^x$ where $a \in \mathbb{R}$.

d. (*6 pts*) $\lim_{x \to 0^+} (\sin x)^{\frac{1}{\ln x}}$.

5. (*15 points*) Determine which of the following functions are uniformly continuous. Justify.

a. (*5 pts*) $f(x) = \frac{x - 1}{x + 1}$ on $(0, 1)$.

b. (*5 pts*) $f(x) = \ln x$ on $(0, 1)$.

c. (*5 pts*) $f(x) = e^x$ on $(-\infty, \infty)$.

6. (*10 points*) Let f be a monotonic function[2] on $[0, 1]$. Let x_0 be a point in $(0, 1)$ and assume that $\lim_{x \to x_0} f(x)$ exists. Show that f is continuous at x_0.

7. (*7 points*) Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

Prove that f is discontinuous at every point.

8. (*6 points*) [**Bonus; no partial credit**] Give an example of a function f defined on $\mathbb{R}$ which is continuous on $\mathbb{Z}$ and discontinuous at every point in $\mathbb{R} \setminus \mathbb{Z}$. Justify.

[2] A function f is monotonically increasing if $f(x) \leq f(y)$ whenever $x \leq y$. Likewise, a function f is monotonically decreasing if $f(x) \geq f(y)$ whenever $x \leq y$. A function f is monotonic if f is monotonically increasing or monotonically decreasing.