1. (10 points) Determine whether the following series converge or diverge. Prove your claims.

(a) (5 points) \[ \sum_{n=1}^{\infty} n^{-1 - \frac{1}{n}} \]

(b) (5 points) \[ \sum_{n=1}^{\infty} \left( n - \sqrt{n^2 - 1} \right) \]

Solution. (a) This series diverges. To see this, we represent the series as

\[ \sum x_n \] where \[ x_n = \frac{1}{n} \cdot \frac{1}{n^{1/n}} \]

and apply the Limit Comparison Theorem with \[ y_n := \frac{1}{n} \]. Since \( \lim n^{1/n} = 1 \), we have \( \lim x_n/y_n = 1 \). Since the harmonic series \( \sum y_n \) diverges, it follows that \( \sum x_n \) diverges as well.

(b) This series diverges, too. Multiplying and dividing by the conjugate, we can represent this series as

\[ \sum x_n \] where \[ x_n = \frac{1}{n + \sqrt{n^2 - 1}} \].

Apply the Limit Comparison Theorem with \( y_n := \frac{1}{n} \). The standard computation shows that \( \lim x_n/y_n = 1/2 \). Since the series \( \sum y_n \) diverges, it follows that \( \sum x_n \) diverges as well.

2. (10 points) Let \( f \) and \( g \) be continuous functions on an interval \( I = [a, b] \). Assume that

\[ \sup\{f(x) : x \in I\} = \sup\{g(x) : x \in I\}. \]

Prove that there exists a point \( c \in I \) such that \( f(c) = g(c) \).
Solution. The Max/Min Theorem shows that the supremum of $f$ on $I$ is attained at some point $x_1 \in I$ and the supremum of $g$ on $I$ is attained at some point $x_2 \in I$. The assumption is that $f(x_1) = g(x_2)$. Consider the function $h := f - g$. We have

$$h(x_1) = f(x_1) - g(x_1) = g(x_2) - g(x_1) \geq 0,$$

where the last inequality follows since $g$ attains its maximum at $x_2$. Similarly we obtain

$$h(x_2) = f(x_2) - g(x_2) = f(x_2) - f(x_1) \leq 0.$$

Since $h = f - g$ is a continuous function, the Intermediate Value Theorem implies the existence of a point $c \in I$ such that $h(c) = 0$ and thus $f(c) = g(c)$.

3. (10 points)

(a) (5 points) Let $g$ be a function that is continuous at 0. Prove that the function $f(x) := xg(x)$ is differentiable at 0.

(b) (5 points) Conversely, let $f$ be a function that is differentiable at 0 and such that $f(0) = 0$. Prove that there exists a function $g$ that is continuous at 0 and satisfies $f(x) = xg(x)$ for all $x$.

Solution. (a) We have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} g(x) = g(c),$$

where the first equality follows from the definition of $f$ and the second, by continuity of $g$ at 0. Thus $f$ is differentiable at 0.

(b) Define the function $g$ as follows:

$$g(x) := \begin{cases} f(x)/x, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then $g$ clearly satisfies $f(x) = xg(x)$ for all $x$. Continuity at 0 holds since

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0).$$
4. \((10 \text{ points})\) Show that the equation \(e^x = 1 - x\) has exactly one real solution.

**Solution.** One solution is \(x = 0\). To prove that there are no other solutions, let us consider the function

\[
f(x) := e^x - 1 + x
\]

and show that \(f(x) \neq 0\) for all \(x \neq 0\). Suppose this does not hold; then \(f(x_0) = f(0) = 0\) for some \(x_0 \neq 0\). By Rolle’s theorem, there exists \(c\) strictly between 0 and \(x_0\) such that \(f'(c) = 0\). However, \(f'(c) = e^c + 1 \geq 1\) for all \(c \in \mathbb{R}\). This contradiction completes the proof.