Midterm Exam 2. Math 451, Fall 2015, Prof. Roman Vershynin

1. *(10 points)* Determine whether the following series converge or diverge. Prove your claims.

(a) *(5 points)* $\displaystyle\sum_{n=1}^{\infty} n^{-1-\frac{1}{n}}$

(b) *(5 points)* $\displaystyle\sum_{n=1}^{\infty} \big(n - \sqrt{n^2-1}\big)$

Solution. (a) This series diverges. To see this, we represent the series as

$$\sum x_n \quad \text{where} \quad x_n = \frac{1}{n} \cdot \frac{1}{n^{1/n}}$$

and apply the Limit Comparison Theorem with $y_n := \frac{1}{n}$. Since $\lim n^{1/n} = 1$, we have $\lim x_n/y_n = 1$. Since the harmonic series $\sum y_n$ diverges, it follows that $\sum x_n$ diverges as well.

(b) This series diverges, too. Multiplying and dividing by the conjugate, we can represent this series as

$$\sum x_n \quad \text{where} \quad x_n = \frac{1}{n + \sqrt{n^2-1}}.$$

Apply the Limit Comparison Theorem with $y_n := \frac{1}{n}$. The standard computation shows that $\lim x_n/y_n = 1/2$. Since the series $\sum y_n$ diverges, it follows that $\sum x_n$ diverges as well.

2. *(10 points)* Let f and g be continuous functions on an interval $I = [a, b]$. Assume that

$$\sup\{f(x) : x \in I\} = \sup\{g(x) : x \in I\}.$$

Prove that there exists a point $c \in I$ such that $f(c) = g(c)$.

Solution. The Max/Min Theorem shows that the supremum of f on I is attained at some point $x_1 \in I$ and the supremum of g on I is attained at some point $x_2 \in I$. The assumption is that

$$f(x_1) = g(x_2).$$

Consider the function $h := f - g$. We have

$$h(x_1) = f(x_1) - g(x_1) = g(x_2) - g(x_1) \geq 0,$$

where the last inequality follows since g attains its maximum at x_2. Similarly we obtain

$$h(x_2) = f(x_2) - g(x_2) = f(x_2) - f(x_1) \leq 0.$$

Since $h = f - g$ is a continuous function, the Intermediate Value Theorem implies the existence of a point $c \in I$ such that $h(c) = 0$ and thus $f(c) = g(c)$.

3. *(10 points)*

(a) *(5 points)* Let g be a function that is continuous at 0. Prove that the function $f(x) := xg(x)$ is differentiable at 0.

(b) *(5 points)* Conversely, let f be a function that is differentiable at 0 and such that $f(0) = 0$. Prove that there exists a function g that is continuous at 0 and satisfies $f(x) = xg(x)$ for all x.

Solution. (a) We have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} g(x) = g(c),$$

where the first equality follows from the definition of f and the second, by continuity of g at 0. Thus f is differentiable at 0.

(b) Define the function g as follows:

$$g(x) := \begin{cases} f(x)/x, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then g clearly satisfies $f(x) = xg(x)$ for all x. Continuity at 0 holds since

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0).$$

4. *(10 points)* Show that the equation $e^x = 1 - x$ has exactly one real solution.

Solution. One solution is $x = 0$. To prove that there are no other solutions, let us consider the function

$$f(x) := e^x - 1 + x$$

and show that $f(x) \neq 0$ for all $x \neq 0$. Suppose this does not hold; then $f(x_0) = f(0) = 0$ for some $x_0 \neq 0$. By Rolle's theorem, there exists c strictly between 0 and x_0 such that $f'(c) = 0$. However, $f'(c) = e^c + 1 \geq 1$ for all $c \in \mathbb{R}$. This contradiction completes the proof.