MATH/STATS 525, Winter 2008

Syllabus:

- OH: fixed
- Prereqs: 451
- Assignments: reading textbook & HW.
- Canvas: communication → email
  Announcements.
- lec. notes posted

Thursday, Jan 5

1.1 – 1.2 Probability Spaces.

Informally:

An experiment with random outcomes is modeled using a probability space (Ω, Σ, P) where:

- Ω is the set of all possible outcomes, the "sample space"
- Σ is the set of events, where each event consists of one or more outcomes;
- P is the assignment of probabilities to events, i.e. a function from events to probabilities

Ex Toss a coin 3 times. What is the probability of getting exactly two heads?

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Σ: all $(2^8 = 256)$ subsets of Ω.

P: assigns $\frac{1}{8}$ to each of the 8 outcomes of Ω.

The event we are interested in is

$$E = \{HHT, HTH, THH\} \in \Sigma.$$

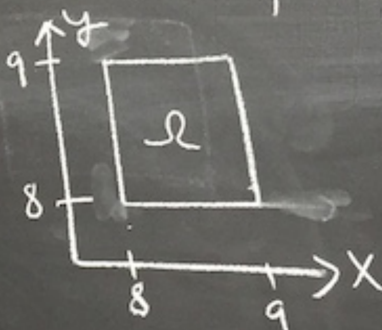
$$P(E) = \frac{|E|}{|\Omega|} = \frac{3}{8}$$

Ex My neighbor and I walk out of our at random times between 8–9 am, independently of each other. (Any time between 8–9 is equally likely.) What is the probability that my neighbor walks out at least 20 min ahead of me?

$\Omega = \{(x, y) : x, y \in [8,9]\}$

x: me, y: neighbor

Ω is infinite.



Σ: all possible subsets of Ω.

P: ? Can't assign meaningful prob. to outcomes (points!)

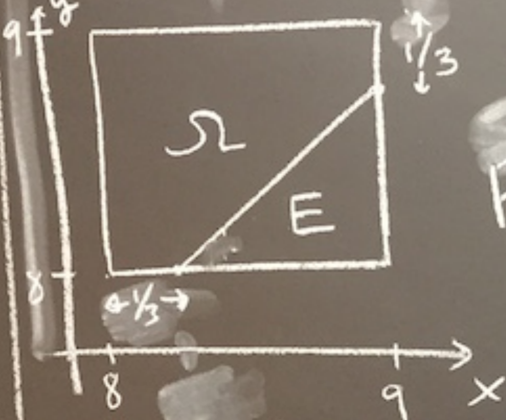
Can assign to events:

$$P(E) := \frac{|E|}{|\Omega|} \quad \text{area}$$

The event we are interested in is

$$E = \{ (x,y) \in \Omega : \ y < x - \tfrac{1}{3} \}$$

$\tfrac{1}{3} = 20/60$



Thus

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{2}}{1} = \frac{2}{9}.$$

<u>Formally</u>: want to make sure natural axioms are satisfied.

<u>Def</u>. Let Ω be (any) set.

A family Σ of subsets of Ω is called a <u>σ-algebra</u> if:

- $\emptyset \in \Sigma$
- $A \in \Sigma$ implies $A^c \in \Sigma$ (Σ "Closed under complement")
- $A_1, A_2, \ldots \in \Sigma$ implies $\bigcup_{i=1}^{\infty} A_i \in \Sigma$.

(Σ "Closed under countable unions")

The sets that form Σ are called <u>events</u>.

nts!)

Ex/ A ...
under...

Def Let ...
A p...
is a...
such...

Def A ...
Ω is a ...
P is a ...

Ex A σ-algebra is automatically closed under countable intersections as well.

Def Let Σ be a σ-algebra on a set Σ.
A probability measure (or simply "probability") is a function $P : \Sigma \to [0,1]$ such that:

- $P(\emptyset) = 0$
- $\forall\, A_1, A_2, \ldots \in \Sigma$ finite or countably infinite sequence of disjoint events,

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i).$$

(P is "countably additive")

Def A probability space is a triple (Ω, Σ, P) where Ω is a set, Σ is a σ-algebra on Ω, and P is a probability on Σ.

$\frac{2}{9}$

al

r complement")

e unions")

ts

Remark (Meaning of set operations)

$\forall$ events A, B;

- $P(A \cup B) = P(A$ <u>or</u> B occur$)$
  (or both)
- $P(A \cap B) = P(A$ <u>and</u> B occur$)$
  (simultaneously)
- $P(A^c) = P(A$ does <u>not</u> occur$)$.

Ex Toss a coin 3 times

A = "getting at least two heads"

B = "getting at least two tails"

$A = \{HHH, HHT, HTH, THH\}$

$B = \{TTT, TTH, THT, HTT\}$

$A \cup B = \Omega$, $A \cap B = \phi$, $A^c = B$.

1.2.

Prop. 1.

(i)

(ii)

(iii)

(iv)

Proof

(i

Su

1.2. Properties of probability.

Prop. 1.9 $\forall$ events A, B;

(i) $P(\Omega) = 1$

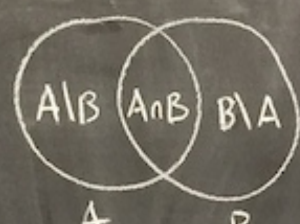
(ii) $P(A^c) = 1 - P(A)$

(iii) If $A \subset B$ then $P(A) \leq P(B)$

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
("inclusion-exclusion principle).

Proof (i)-(iii) — exercise; see book.

(iv):



Scizophr, Bipolar
1.2%. 2.6%.
S&B : 0.8%,
$P(S \text{ or } B) = \frac{1.2+2.6-0.8}{100}$
$= 3\%$.

By countable additivity,

$P(A) = P(A \setminus B) + P(A \cap B)$,

$P(B) = P(B \setminus A) + P(A \cap B)$,

$P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B)$.

Sum up $\Rightarrow$ QED. $\square$.

Ex 1.2% of adults is diagnosed with schizophrenia,
2.6% with bipolar disorder,
0.8% with both.
What % has either S or B?

$$P(S \text{ or } B) = \frac{1.2 + 2.6 - 0.8}{100} = \frac{3}{100} \quad \text{Ans: } 3\%$$

Remark: $\exists$ general incl-excl. principle for more than 2 events

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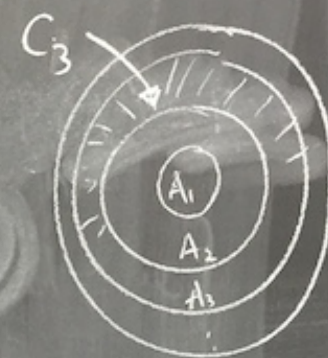
Prop 1.10 (Continuity)

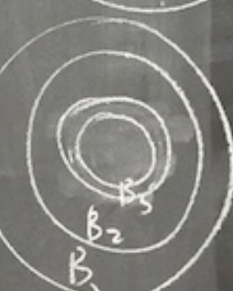
(i) $\forall$ events $A_1 \subset A_2 \subset A_3 \subset \ldots$,

$$\lim_{i\to\infty} P(A_i) = P(A)$$
where $A = \bigcup_{i=1}^{\infty} A_i$



(ii) $\forall$ events $B_1 \supset B_2 \supset \ldots$,

$$\lim_{i\to\infty} P(B_i) = P(B)$$
where $B = \bigcap_{i=1}^{\infty} B_i$



Proof (i): Let $C_i := A_i \setminus A_{i-1}$. Then:

$$A_n = \bigcup_{i=1}^{n} C_i \quad , \quad A = \bigcup_{i=1}^{\infty} C_i \qquad \text{(Check!)}$$

Using countable additivity,

$$P(A) = P\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} P(C_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(C_i) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} C_i\right)$$

$$= \lim_{n\to\infty} P(A_n). \quad \text{QED}$$

(ii) - Ex

1.3. Examples

Ex 1. n people, including Bob & Alice, are seated in a row. What is prob. that Bob is to the right of Alice?

$$P(B \text{ to the right of } A) = P(A \text{ to the right of } B)$$

same kind of event, up to the identities

The sum of the two prob's $= 1$

$\Rightarrow$ each $= \frac{1}{2}$

("Symmetry argument")

Ex. 2 What is the prob. that Bob is next right to Alice?

\# of all possible seating arrangements $= n!$

\# of arrangements where B is next right to A $= (n-1)!$

(replace A & B by one object AB, arrange $n-1$ objects)

$$\Rightarrow P(B \text{ next right to } A) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

Ex: $P(A, B \text{ sit next to each other}) = ?$

Remark In Ex 2, we used the following property:

$$\boxed{\text{If } \Omega \text{ is finite, and all outcomes are equally likely, then } P(A) = \frac{|A|}{|\Omega|} \quad \forall A \subset \Omega}$$

Proof Let $\Omega = \{\omega_1, \dots, \omega_n\}$ ← outcomes

$$1 = P(\Omega) = \sum_{i=1}^{n} \underbrace{P(\{\omega_i\})}_{\text{the same}} = n \cdot P(\{\omega_i\})$$

$$\Rightarrow P(\{\omega_i\}) = \frac{1}{n}.$$

Then $\forall A \subset \Omega$,

$$P(A) = \underbrace{\sum_{i:\, \omega_i \in A}}_{|A| \text{ terms}} \underbrace{P(\{\omega_i\})}_{= 1/n} \quad \text{(additivity)}$$

$$= \frac{|A|}{n}. \qquad \square$$