MATH/STATS 525, Winter 2008

Syllabus:

- OH: fixed
- Prereqs: 451
- Assignments: reading textbook & HW
- Canvas: communication → email Announcements
- Lec. notes posted
Informally:

An experiment with random outcomes is modeled using a probability space \((\Omega, \Sigma, P)\) where:

- \(\Omega\) is the set of all possible outcomes, the "sample space"
- \(\Sigma\) is the set of events, where each event consists of one or more outcomes,
- \(P\) is the assignment of probabilities to events, i.e., a function from events to probabilities.

**Ex:** Toss a coin 3 times. What is the probability of getting exactly two heads?

\(\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}\)

\(\Sigma\) is all \(2^8 = 256\) subsets of \(\Omega\).

\(P\) assigns \(\frac{3}{8}\) to each of the 8 outcomes of \(\Omega\).
The event we are interested in is:

\[ E = \{HHT, HTH, TTH\} \in \Sigma. \]

\[ P(E) = \frac{|E|}{|\Omega|} = \frac{3}{8}. \]

My neighbor and I walk out of our at random times between 8–9 am, independently of each other.

(Any time between 8–9 is equally likely.)

What is the probability that my neighbor walks out at least 20 min ahead of me?

\[ \Omega = \{(x,y) : x, y \in [8,9]\} \]

\[ \text{me} \quad \text{neighbor} \quad \Omega \text{ is infinite}. \]

\[ \Sigma : \text{all possible subsets of } \Omega. \]

\[ P : \text{Can't assign meaningful prob. to outcomes (points!)} \]

Can assign to events:

\[ P(E) := \frac{|E|}{|\Omega|} \sum \text{area}. \]
The event we are interested in is:

\[ E = \{(x, y) \in \mathbb{R}^2 : y < x - \frac{1}{3} \} \]

Thus:

\[ P(E) = \frac{|E|}{|\mathbb{R}^2|} = \frac{\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}}{1} = \frac{2}{9}. \]

Formally: want to make sure natural axioms are satisfied.

**Def.** Let \( \Omega \) be any set.

A family \( \Sigma \) of subsets of \( \Omega \) is called a \( \sigma \)-algebra if:

1. \( \emptyset \in \Sigma \)
2. \( A \in \Sigma \) implies \( A^c \in \Sigma \) (\( \Sigma \) "closed under complement")
3. \( A_1, A_2, \ldots \in \Sigma \) implies \( \bigcup_{i=1}^{\infty} A_i \in \Sigma \) (\( \Sigma \) "closed under countable unions")

The sets that form \( \Sigma \) are called events.
Ex: A $\sigma$-algebra is automatically closed under countable intersections as well.

Def: Let $\Sigma$ be a $\sigma$-algebra on a set $\Omega$. A probability measure (or simply "probability") is a function $P : \Sigma \rightarrow [0, 1]$ such that:

- $P(\emptyset) = 0$
- $\forall A_1, A_2, \ldots \in \Sigma$ finite or countably infinite sequence of disjoint events,

$$P(\bigcup A_i) = \sum P(A_i)$$

($P$ is "countably additive")

Def: A probability space is a triple $(\Omega, \Sigma, P)$ where $\Omega$ is a set, $\Sigma$ is a $\sigma$-algebra on $\Omega$, and $P$ is a probability on $\Sigma$. 

(\text{Def: complement})

(\text{Def: disjoint sets})

(\text{Def: unions})

(\text{Def: equal sets})
Remark (Meaning of set operations)

\[ \forall \text{ events } A, B; \]
\[ \cdot P(A \cup B) = P(A \text{ or } B \text{ occur}) \]
\[ \cdot P(A \cap B) = P(A \text{ and } B \text{ occur}) \]
\[ \cdot P(A^c) = P(A \text{ do not occur}) \]

Ex

Toss a coin 3 times

\[ A = \text{ "getting at least two heads"} \]
\[ B = \text{ "getting at least two tails"} \]
\[ A = \{HHH, HHT, HTH, THH\} \]
\[ B = \{TTT, TTH, THT, HTT\} \]
\[ A \cup B = \Omega, \quad A \cap B = \emptyset, \quad A^c = B \]
1.2. Properties of probability.

Prop. 1.9. All events $A, B,$

(i) $P(\Omega) = 1$

(ii) $P(A^c) = 1 - P(A)$

(iii) If $A \subset B$ then $P(A) \leq P(B)$

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

("inclusion-exclusion principle").

Proof. (i) - (iii) - exercise; see book.

(iv):

By countable additivity,

$P(A) = P(A \setminus B) + P(A \cap B),$

$P(B) = P(B \setminus A) + P(A \cap B),$ 

$P(A \cup B) = P(A \setminus B) + P(B \setminus A) + P(A \cap B).$

Sum up $\Rightarrow$ QED.
Ex. 1.2% of adults is diagnosed with schizophrenia, 0.8% with both.

What % has either S or B?

\[
\Gamma (S \text{ or } B) = \frac{1.2 + 2.6 - 0.8}{100} = \frac{3}{100} \quad \text{Ans: } 3\%
\]

Remark: For general ind-excl. principle for more than 2 events.

Prop 1.10 (Continuity)

(i) All events \( A_1 < A_2 < A_3 < \cdots \),

\[
\lim_{i \to \infty} P(A_i) = P(A) \quad \text{where } A = \bigcup_{i=1}^{\infty} A_i
\]

(ii) All events \( B_1 > B_2 > \cdots \),

\[
\lim_{i \to \infty} P(B_i) = P(B) \quad \text{where } B = \bigcap_{i=1}^{\infty} B_i
\]

Proof (i): Let \( C_i = A_i \setminus A_{i+1} \). Then:

\[
A_n = \bigcup_{i=1}^{n} C_i \quad A = \bigcup_{i=1}^{\infty} C_i \quad \text{(Check!)}
\]

Using countable additivity,

\[
P(A) = P\left( \bigcup_{i=1}^{\infty} C_i \right) = \sum_{i=1}^{\infty} P(C_i) = \lim_{n \to \infty} \sum_{i=1}^{n} P(C_i) = \lim_{n \to \infty} P\left( \bigcup_{i=1}^{n} C_i \right)
\]

\[
= \lim_{n \to \infty} P(A_n) \quad \text{QED.}
\]
1.3. Examples

Ex 1. n people, including Bob & Alice, are seated in a row. What is prob. that Bob is to the right of Alice?

\[ P(\text{B to the right of A}) = P(\text{A to the right of B}) \]

- Same kind of event, up to the identities
- The sum of the two probs = 1
  - \( \Rightarrow \) each = \( \frac{1}{2} \)

("Symmetry argument")

Ex 2. What is the prob. that Bob is next right to Alice?

\[ \# \text{ of all possible seating arrangements} = n! \]
\[ \# \text{ of arrangements where B is next right to A} = (n-1)! \]
(replace A & B by one object AB, arrange n-1 objects)

\[ \Rightarrow P(\text{B next right to A}) = \frac{(n-1)!}{n!} = \frac{1}{n} \]

Ex: \( P(A, B \text{ isr next to each other}) = \)
Remark: In Ex 2, we used the following property:

If $\Omega$ is finite, and all outcomes are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|} \quad \text{for } A \subseteq \Omega$$

Proof: Let $\Omega = \{\omega_1, \ldots, \omega_n\}$

$$1 = P(\Omega) = \sum_{i=1}^{n} P(\{\omega_i\}) = n \cdot P(\{\omega_1\})$$

Then $\forall A \subseteq \Omega$,

$$P(A) = \sum_{\omega_i \in A} P(\{\omega_i\}) \quad (\text{additivity})$$

$$= \frac{|A|}{n} \quad \text{(as the outcomes are equally likely)}$$